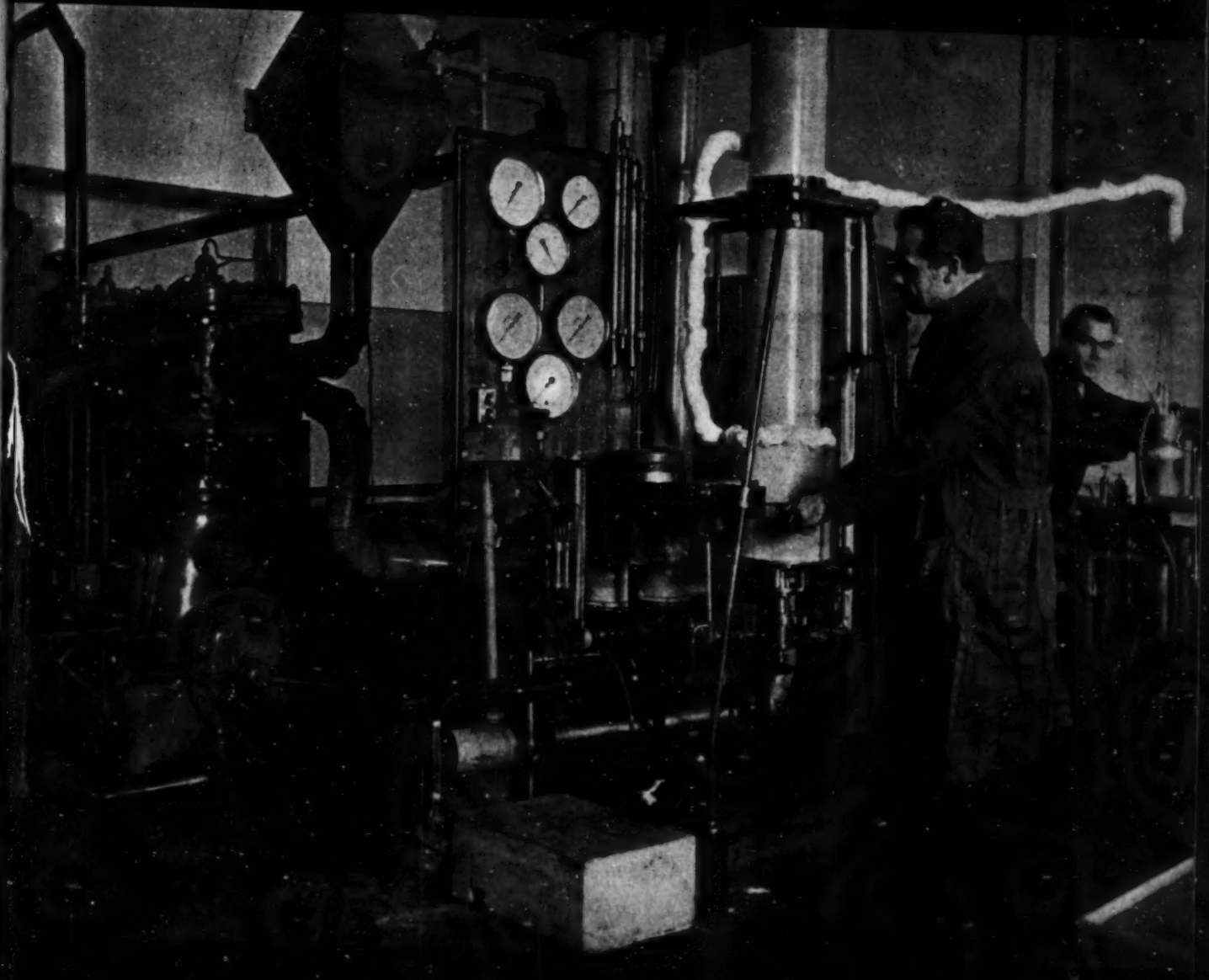
JOURNAL OF APPLIED PHYSICS



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Volume 13 August, 1942 Number 8

Manpower in Physics — A National Emergency

IN the second report[1] of the War Policy Committee of the American Institute of Physics, issued July 8, 1942, emphasis is placed upon the alarmingly serious situation of national manpower in physics. The Institute's Committee believes that this situation has already reached the proportions of a national emergency which may be summarized as follows:

1. The design, production, operation, and maintenance of new physical instruments of warfare are *essential* to the successful prosecution of this war. (Examples: submarine and aircraft location, improvement of anti-aircraft fire, automatic fuses, signaling devices, magnetic mines, etc.)
2. The need for more physicists for these purposes is large and urgent.
3. The number of physicists in this country is small, only about 7000, and a substantial portion of these are already engaged in direct war work, leaving altogether too few to provide physics training for Army and Navy personnel and those needed for war research and production.
4. Training of physicists is not an easy or short-time process.
5. Unless prompt, effective measures are taken the shortage of physicists will be disastrously acute and no adequate program for training new physicists can be effected.

It is undoubtedly true that the Army, the Navy, and the War Manpower Commission also are beginning to recognize the existence of this emergency situation in physics. To meet it, the War Policy Committee is urging that these groups take the following steps:

1. Arrange for teacher training to provide for the very great amount of physics teaching which will be needed, not only in producing physicists, but in connection with training programs of the Army and Navy in which it is essential to convey some knowledge of physics to over 200,000 men and women within a year.
2. Revise the situation of physicists with respect to Selective Service so as to assure students and teachers of physics the possibility of continuing their work without uncertainty.
3. Provide loans, scholarships, or other assistance to well-qualified students who need aid to continue their training in physics.
4. Any men in the Army and Navy with physics training who are not actually employing that training in their work should be transferred to positions where physicists are now urgently needed, either by detail or by discharge.
5. Start a public relations program as to the meaning of physics and its importance in the war, this being necessary to secure public approval and understanding of the necessity of the preceding four steps.

Already the teaching loads in universities have reached tremendous proportions. Example after example is being provided showing that present summer session enrollments in physics are three, four, and even five times the normal summer enrollment. Thus there is ample evidence of the need for action of the type which the Institute's Committee has recommended. What the situation in universities and colleges is going to be during the next academic year is difficult to predict. A statement of the teaching problems as they appear to the War Policy Committee is reprinted on page 477 of this issue.

[1] The first report of this committee was published in J. App. Phys. **13**, 345–347 (1942).

Science In The U. S. S. R.*

J. G. CROWTHER

Mr. J. G. Crowther has an intimate knowledge of Soviet science and scientists, numbering many of the latter among his personal friends. He is thus particularly well qualified to write with authority upon the personalities, ideas, and achievements of the present-day successors of Lomonosov and Mendeléeff. The freshness and vigor of Russian thought are vividly illustrated by Rehbinder's views that science should consciously aim at creating new industries and not be content with improving old ones.

SINCE the time of Peter the Great, there has always been notable scientific research in Russia. Peter founded the Academy of Sciences in 1724, after the French model. He and his successors depended a good deal for their academicians on importation from abroad. Daniel Bernoulli went to St. Petersburg in that year, and shortly afterwards he was followed by Leonhard Euler. The Academy from its foundation had a very high standard, but from the nature of its origin it was an exotic intellectual ornament rather than a working organ of the state. In the last ten years especially it has been fundamentally reorganized. The principle of the Academy during the first two centuries of its existence was the appointment of men of ability, the more gifted the better, with handsome salaries and light duties. They were then expected to follow the promptings of their inspiration, and embellish history with new knowledge.

The reorganized Academy has quite a different principle. It is intended to be a powerful organ in the advancement of human life. It has the duty of planning scientific activity in the U.S.S.R. so that it will achieve that end most effectively. The Academy was transferred to Moscow so that its directorate would have closer contact with the Government and its institutes would be in a safer strategical location.

The idea of planning is to develop scientific research in a balanced manner, discover and encourage gifted individuals, and lay down lines of investigation consonant with the main interests of the population. For instance, planning is intended to prevent overlapping and stampedes into fashionable subjects. It will see that there is a proper division of attention between, say, nuclear physics and radiophysics. The periodical reports on the activities of all laboratories help to prevent good men from being overlooked. The coordination of the general lines of research with the main interests of the population taps a strong stream of intellectual energy that might have been dissipated in other directions.

The Physico-Technical Institute directed by Professor A. F. Joffe at Leningrad is the most famous of its kind in the U.S.S.R. It is one of the group of institutes attached to the Academy of Sciences, and its activities are a part of the Academy's general plan of research. It was started by Joffe in September, 1918 in a few rooms of the old Polytechnic Institute. It was moved in 1923 into a handsome building which had been started in 1914 as a home for retired engineers, but never completed.

As the Institute grew, large blocks of new laboratories and machine shops were added to it. The original staff consisted of Joffe, Semenov, Tchernitchev, Dorfmann, Lukirsky, and Frenkel. The first aim of the Institute was the training of physicists who could become directors of new institutes and foci of new growths of science in distant parts of the country. As a result of this plan, physical institutes have been founded in Kharkov, Dnepropetrovsk, Tomsk, Sverdlovsk, and Samarkand.

This development sprang from Joffe's insti-

* This article is being reprinted with the generous permission of *Endeavour*. The journal, *Endeavour*, is a new quarterly review, published in England by Imperial Chemical Industries, whose stated purpose is to carry overseas news of the continuing vitality and progress of the sciences in these days when not only nations but the internationalism of the sciences are threatened.

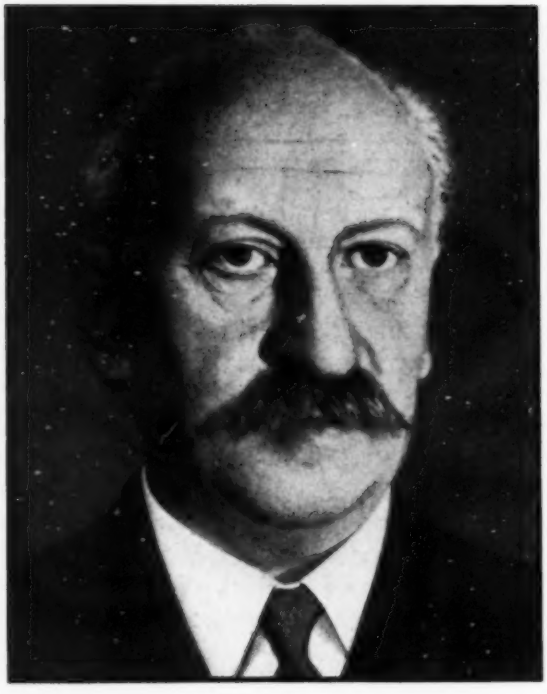
tute. He is accordingly admired as the founder of modern physics in the U.S.S.R. His sixtieth birthday was celebrated recently by a fine volume in the Academy of Sciences' *Journal of Physics*, which consisted exclusively of a collection of papers by his former pupils.

Joffe's own researches in recent years have been on semiconductors. He regards these as substances in which electrons have a moderate degree of freedom, in contrast with insulators in which they have very little, and in metals in which they have a great deal. Their study throws light on the mechanism of the breakdown of insulators, because this happens too suddenly for direct investigation. On the other hand, it cannot be deduced from the properties of metals because the flow of current starts too easily. Another line pursued by Joffe is the strength of solids. It is found that the theoretical strength calculated from Born's theory is several hundred times the observed strength. In his researches in this problem, Joffe and his colleagues observed that the tensile strength of a crystal of rocksalt was increased 20 times while immersed in hot water. He attributed this to the removal by solution of cracks on the surface of the crystal, which, according to the theory of Griffith, are a major cause of the observed weakness of glass and other solids.

Stepanow's suggestion that friction between adjacent planes in a crystal under strain leads to a momentary liquefaction has arisen out of Joffe's work, and has helped to explain why a piece of metal may be suddenly twisted and remain strongly set in its new shape. The metal at the surfaces at the planes of slip is momentarily melted, and then sets quickly while the piece is in its new shape.

The recent investigations of S. E. Bresler and P. A. Phinogenov on microscopic steel balls

The Institute of Physical Problems, Moscow.



Professor A. F. Joffe.

for bearings is a characteristic product of the Joffe school. It follows from a formula for the deformation of spheres first given by Hertz that small balls may be as effective in bearings as large ones, because there are more of them. Accordingly, mass production methods of making microscopic steel balls were investigated. Soft carbon steel wire was blown through an electric arc into an atmosphere of nitrogen, producing a "fog" of steel particles in it. When examined, these were found to be shiny and from a few thousandths to a few tenths of a millimeter in diameter. The larger ones were prolate, and deviated by 5 to 10 percent from the spherical; while the smaller ones, owing to the more powerful effect of surface tension, were nearly perfect. Balls about 1/10 mm in diameter were separated by sieves. When mixed with oil they reduced friction 2½ times as much as oil used alone.

Perhaps the most remarkable achievement that has come from Joffe's institute is Skobeltzyn's discovery of the very swift particles in cosmic rays. In 1926 he began a study of the velocity of electrons ejected from radioactive sources by photons, with the aid of a Wilson chamber surrounded by a magnetic field. He published photographs in 1927 of particles which shot through the field with such energy that their curvature could not be detected.

The apparatus with which he secured this fundamental observation was of the most provisional character. The timing of the apparatus was controlled by an old Atwood's machine, which seems to be about the only valuable application of this dreary invention on record.

Skobeltzyn obtained several photographs of positrons, but was not lucky enough to be the first to recognize their nature. He has recently been working on the components of the cosmic radiation, and has argued against the theory of Heisenberg and Euler that the spontaneous disintegration of mesotrons is the mechanism responsible for the formation of the electronic component observed in the lower layers of the atmosphere, because, in his opinion, it fails to explain the observed divergence between the curves describing the increase in the intensities of the hard and the soft components with height.

One of the most striking figures in Soviet science is N. N. Semenov. He is indeed one of the most remarkable men in the world, and is the chief founder of the contemporary theory of chain reactions.

Semenov and Kapitza are old friends and fellow students. In the Leningrad flat of the late Mme. Kapitza, the scientist's mother, there used to be a beautiful oil painting of the pair in discussion over an x-ray tube. It is to be hoped that this historic picture is safe. Semenov started research as a physicist, and studies of the mechanism of the breakdown of insulation, in which a progressive change proceeds through the material, led him to the study of chain reactions. The classical experiment which accelerated his school on its brilliant career was made by Chariton and Walta in 1926. Berthollet had observed in 1797 that phosphorus and oxygen, in spite of their intense attraction for each other at ordinary pressures, do not combine if the pressure of the oxygen exceeds a certain higher limit. Joubert, in 1874, added that the reaction also stops if the pressure falls below a certain lower limit. Chariton and Walta found that if argon were introduced into the vessel,

the upper pressure limit was lowered. Semenov showed that the phenomenon could be explained by the theory of chain reactions. Many more discoveries in the same field have come from his institute. This was founded as a separate laboratory in 1931, an old secondary school building being adapted to the purpose.

Chariton is an old Cavendish student, though his main achievements have been accomplished in chemical physics. He has recently published a characteristic paper on the inability of small charges of explosives to detonate. He made small calibrated glass tubes which were closed at one end, the other end opening out into a funnel about 1 cm in diameter at the top. These were filled with nitroglycerine, which was left in contact with a priming charge of lead azide at the open end. It was found that if the diameter of the tube was less than 2.08 mm the detonation did not pass from the wide funnel-shaped part of the tube into the narrower cylindrical part. With a 12-percent solution of methyl ether in nitroglycerine, the limit rose to about 3.6 mm. One may suppose that, below a certain diameter, the duration of the scattering of the outer part of the charge is less than the duration of the reaction, so that detonation does not occur.

The Semenov school has contributed greatly to the strength of the Soviet Union, besides the advancement of knowledge, because its studies on chain reactions have provided experts on "knock" in engines, explosives, and other things of practical importance.

Peter Kapitza is well known to many British scientists. The handsome Institute of Physical Problems has been specially built for him in Moscow. He has recently invented a turbine air-liquefier, in which the initial compression of the air is only 5 atmospheres, instead of 200 atmospheres as in the Linde process. This is achieved by taking the energy out of the gas by making it do work in the turbine, instead of depending on the Joule-Thomson effect, which is of secondary magnitude. As the temperature of the air flowing through the turbine, unlike that of the stream in a steam turbine, is low, the working fluid is much more dense, being about five times that of steam at 250°C. Consequently, the turbine has to be designed to withstand considerable centrifugal forces, like a water turbine.

Professor Peter Kapitza.

Air is compressed by a 50-hp compressor, and after passing through water and air coolers is fed at about 7 atmospheres into the turbine. This runs at 40,000 r.p.m., and develops about 4 hp. The pressure of the air drops 75 percent in its passage through the turbine, and if it enters at about −158°C it emerges at about −187°C, the boiling point of oxygen.

The present apparatus produces about 30 kg of liquid air per hour, and with minor improvements should be as efficient as ordinary high pressure liquefiers. It is compact, owing to the absence of auxiliary equipment, and it delivers liquid air within the short period of twenty minutes from starting.

Another interesting scientist is P. Rehbinder, who directs research on the chemistry and physics of surfaces in the Physical Institute of the Academy of Sciences in Moscow. He has a burning enthusiasm and energy for research, and original views on the role of science in society. He considers that science should not be the handmaid of industry, but should consciously aim at the creation of new industries. These are

Scientific education: a chemical laboratory at a university center in Daghestan.

vastly more valuable than trifling improvements in conventional manufacturing processes.

Rehbinder's researches include striking investigations on the phenomena of hardness. He has determined the hardness of specimens by deducing the strengths of the molecular fields at their surface from measurements of wettability, and he finds that his results agree with the usual methods of mechanical scratching. His method has the advantage of not damaging the specimen.

He found that thin films are often much harder than solid blocks of the same material. For instance, a thin film of lead on glass may be twice as hard as glass itself, though block glass is thirty times as hard as block lead. In contrast with this, the application of other substances adsorbed by the surface may reduce the hardness.

Rehbinder has recently been examining the effect of the addition of surface-active substances to the mechanical properties of soap curds. He has added homologues of a number of saturated alcohols, phenol, aniline, etc., to the curds of metallic soaps. He has found that the resistance of the curds to shear is reduced. He believes that the effect is partly analogous to the one he has already described, in which hardness of solids is reduced by adsorbed layers. He thinks that it should be possible to alter the mechanical properties of hard commercial soaps considerably by adding surface-active substances to them during manufacture and use.

Perhaps the most gifted physicist in the U.S.S.R. is L. I. Mandelstamm, whose sixtieth birthday, like Joffe's, was recently celebrated by a special volume in the *Journal of Physics* of papers by his pupils. He works in Moscow University and the Physical Institute of the Academy. Mandelstamm is modest, retiring, and self-critical. He does not like lecturing, and is cautious in publication. He discovered, in collaboration with Landsberg, the effect which now bears the name of Raman, in February, 1928. While they were checking their results, Raman independently discovered the same effect and promptly published his observations in March, 1928. This is only part of Mandelstamm's work. He has created a school of research in non-linear oscillations which is of the highest importance in radiophysics and aerodynamics, and helps to explain the strength of the Soviet Union in these fields.

The theory of non-linear oscillations was founded a century ago in connection with planetary perturbations, and was developed by Poincaré: Mandelstamm began the application of the theory of physical problems about fifteen years ago. Besides leading to improvements in radio circuits, it has inspired the invention of a new type of dynamo. It consists of a condenser which is placed in a circuit containing a self-inductance. The plates of the condenser are shaken by a mechanical motor, so that the distance between them is always changing. This varies the capacity of the condenser and produces a current in the circuit whose period is one-half that of the mechanical oscillator. Thus the energy of the mechanical oscillator is converted into a high voltage electric current.

Such are a few of the Soviet scientists and their researches. There are hundreds of others, in very different branches of science. They are an example of the quality and energy of the scientific workers in the U.S.S.R.

An Estimate of the Teaching Problems in Colleges and Universities for 1942–43 as Made by the War Policy Committee of the American Institute of Physics

(Report of July 8, 1942)

"THE Committee has studied plans of the Army and the Navy to enlist a large fraction of next year's college and university students in training programs to be carried on at the institutions. The programs include the Army Enlisted Reserve and the Navy V-1 program. The Services rely on these programs to provide large pools of officer material and can be expected to make every effort to recruit them to the necessarily large enrollments to meet the essential needs. The Navy program requires a substantial course in physics; the Army program is such as to increase above normal the number of students electing physics. On the basis of joint announcements of the Services and conferences with their representatives, the Committee has estimated that the teaching load in college grade physics will in 1942–43 be from two to three times the highest ever before sustained. This teaching load will fall on faculties necessarily depleted for war research and, in some instances, wastefully dissipated through the direct action or indirect influence of the Selective Service Act and the lack of knowledge in some local boards as to the importance of physics and its wartime role.

"The Committee welcomes the very specific directives which have been issued by Selective Service Headquarters, but urges that high officials of the Army and the Navy issue statements emphasizing the necessity for students and teachers to continue their present work and affirm that in so doing the individuals concerned are performing the highest and most patriotic duty open to them. Such public statements, as well as occupational deferment policies, should particularly include graduate students, since these students are already contributing heavily to the teaching of physics and are being drawn more and more into war research.

"Whatever actions may be taken by Federal authorities (and some are known to be in preparation) designed to lessen the prospective shortage of teachers, the Committee is convinced that they cannot be wholly adequate. The Committee, therefore, urges college and university administrations and faculties to use every expedient they can devise to prepare for the coming flood of enrollments in physics courses. This advice is all the more urgent in view of the fact that the war research programs must continue to expand and that additional faculty members will have to be called away from their teaching duties."

Response by Dr. William D. Coolidge in Accepting the Duddell Medal at the Meeting of the American Physical Society, May 1, 1942

YOUR EXCELLENCY, MR. PRESIDENT, MEMBERS OF THE AMERICAN PHYSICAL SOCIETY, LADIES AND GENTLEMEN:

MAY I first thank Your Excellency for your great courtesy in coming here and giving me the pleasure of receiving this honor at your distinguished hands. This is generous compensation for the disappointment which I naturally felt because present conditions made it inadvisable for me to attend a meeting of The Physical Society in London to receive the award directly from their president.

I am deeply conscious and appreciative of the honor of being made a Duddell Medalist, and for several reasons. The medal takes high luster from the great scientific achievements of Duddell, in whose honor The Physical Society established the memorial fund—a luster enhanced by the eminence of that society itself and by the conspicuous merit of former recipients. It carries special significance and value for me in coming from England, the home of my ancestors, the source of many great achievements in science, and the stalwart guardian of what are to me the finer things of life.

To return for a moment to the work of Duddell. Had he done nothing more than develop the magnetic oscillograph, his contribution to the electrical science and art would have been outstanding. One cannot visit a laboratory today where electrical researches or electrical engineering developments are in progress without being greatly impressed by the importance of the role played by the oscillograph which yields so much essential information obtainable in no other way. For frequencies within its range, the magnetic oscillograph, substantially as developed by Duddell, is still in general use. For higher frequencies, the cathode-ray oscillograph is used, but here also credit must be given to Duddell, for his magnetic oscillograph served to blaze the trail. The art of radio also is heavily in debt to Duddell for his discovery of the singing arc, his design of a high frequency alternator, and of his thermoammeter, all of which contributed to the early development of wireless telegraphy.

To make clear the limitations of my part in the work on which the award was based, I must briefly outline the conditions under which the work was done.

Our laboratory, at the time the ductile tungsten work was started, was only five years old. This laboratory sprang from the vision of Dr. E. W. Rice and his associates. He was then in charge of General Electric engineering and had seen clearly the extent to which the products of our factories had been derived from the results of scientific research, such as had been carried out notably by English physicists.

The laboratory was started in the hope that it might contribute new facts and principles on which new products could be based. It was one of the earliest industrial research laboratories in this country and was established primarily for fundamental research. It was itself an experiment, but an experiment carried out under extremely favorable conditions, as both its sponsors and its director, Dr. W. R. Whitney, were men of great courage, optimism, and faith in the power of scientific research methods. The strongest single psychological factor conducing to the success of the laboratory was the spirit of cooperation which Dr. Whitney was able to build up not only within the laboratory but also between the staff and the engineers of the Company.

That cooperation was so close and so generous and so effective that a very large share of the credit for the ductile tungsten and x-ray work belongs to those who were associated with me in that work. Credit is also due to the courageous and liberal management of the Company for giving its continued support to such work as the ductile tungsten effort, which over a long period of years was exceedingly expensive and of questionable promise.

While the factory process for making ductile tungsten is today fundamentally the same as that developed in the laboratory, it has undergone constant improvement and been given increased utility by our engineers for the various applications which have been found for the product.

Although this process did not constitute the first application of powder metallurgy, it seems to have been at the time the most important one which had been made, and, for this reason it may perhaps be fairly credited with having been in no small measure responsible for the subsequent development of the art of powder metallurgy now widely applied to other metals, and to mixtures of metals and refractory compounds.

My participation in the work on x-rays, which forms the other basis for the award, followed as a result of the keen interest which I had felt in x-rays ever since the announcement of their discovery, also from the work which we did in adapting wrought tungsten to use as a target in the earlier x-ray tube in place of platinum, and from Dr. Irving Langmuir's studies on the thermionic emission of electrons from tungsten filaments in high vacuum. I had the advantage also of intimate firsthand knowledge of the limitations of the x-ray tube of that time and of the source of those limitations. Stable and controllable thermionic electron emission rendered unnecessary the gas content which alone had been responsible for the most troublesome limitations of the earlier tube. At the same time it provided the much desired flexibility of control and operation. I was also fortunate in having readily available not only tungsten wire for the cathode filament but also large masses of wrought tungsten which were at the time indispensable for the target. For when our first successful tubes were made, we did not know that it would ever be possible to degas sufficiently a composite target of copper and tungsten such as had been generally used in the earlier type of tube.

I cannot forego this opportunity to pay my tribute to Sir Owen Richardson who, more than any other, had laid the groundwork for our knowledge of thermionic emission.

From the beginning, it was clear that the high voltage generating equipment which had been developed for use with the earlier tube was ill-suited to the hot cathode type and for this reason later developments of the hot cathode tube and of the high voltage source have, even up to the present time, gone hand in hand.

While some hot cathode tubes of the first so-called universal type are still in use, this design has generally been supplanted by special tubes giving greatly improved performance for various specific uses.

For example, types have been developed for operation on unrectified alternating current. The characteristics of the tube have permitted various forms of equipment in which the tube and the entire high voltage circuit are completely enclosed within grounded metal, thus eliminating all danger of electric shock and facilitating the attainment of adequate x-ray protection. For radiography, definition has been greatly improved in various ways, notably by the use of elongated focal spots and more recently, by this combined with rapid rotation of the target.

For medical therapy and for industrial radiography, tubes for higher voltages than could be used with the first design have been developed, first by merely increasing dimensions. Then, because of troublesome field currents arising from too high potential gradients at the cathode, we were led to depart from the simple two-electrode structure and to use a multiplicity of tubular accelerating electrodes in addition to cathode and anode. This apparently makes it possible to provide tubes for voltages as high as can be produced.

A recent development in the generation of high voltage to supply such tubes is the resonance transformer. This has no iron within the primary and secondary coils, thus permitting the x-ray tube to be located here, where it is electrostatically shielded by the surrounding coils. The size and weight of such equipment have been greatly reduced by substituting for oil, a compressed gas having high dielectric strength.

For x-ray diffraction work, the recent development of tubes with linear focus combined with metallic beryllium windows results on the one hand in great reduction of the required exposure time, and on the other in a considerable extension, in the long wave-length direction of the available spectrum.

It is exceedingly pleasant to think of any useful purpose subserved by war. World War No. I did greatly accelerate both the development and the introduction to use of the hot cathode tube. The present war has already led to the extensive use of the photoroentgenographic method for the chest examination of recruits for the armed forces and this, because of its greatly reduced cost, excellent results, and convenience, is certain to find

widespread use after the emergency. The present war is also stimulating enormously the application of x-rays to industrial uses, both radiographic and fluoroscopic. As an illustration of this, although the first portable million-volt radiographic x-ray outfit went into use only about a year ago, the fortieth one will soon have been completed. These million-volt outfits make possible the rapid examination of even as difficult subjects as steel castings and welds having a thickness of as much as eight inches, and thereby are contributing most helpfully to many industrial war activities. As another example, fluoroscopic outfits are being built and used for the rapid examination of small aluminum die castings, thus permitting immediate rejection of many whose defects could otherwise be seen only after a considerable amount of machine work had been performed on them.

To me, Your Excellency, the greatest value of this award lies in the fact that it is one of the many evidences of the cordial and friendly spirit which has for years existed between British and American scientists. This has been evidenced anew by the close and effective cooperation in the present war work between the scientists of the British Commonwealth and those of this country. It has been greatly facilitated through your various war agencies in England, the National Research Council of Canada, and our Office of Scientific Research and Development. Your scientists are bringing us full reports of the vast amount of splendid work which you are doing in support of the war effort. We are making good use of it and are in turn keeping you informed concerning our work. This augurs well, both for the emergency and for the more leisurely pursuits to follow. This mutual aid in our common cause is strengthening the bonds between our nations, and we may hope that, when victory is won, those bonds may serve to hold us together in the task of rehabilitating the world on a firm basis of liberty, security, and opportunity for all.

In every country of the world there are men of good will, who desire peace and friendly relations with other nations. Misrepresentative governments may gain temporary ascendancy, and by ruthless aggression and acts of cruelty bring upon themselves the hatred of their victims, but there remains, though temporarily submerged and silenced, the peace-loving mass of human beings who comprise the bulk of the nation, and who will respond in kind to an approach in the spirit of friendship and mutual helpfulness.

It is a far cry today, but we must certainly look hopefully to the time, and labor diligently for it, when such relations of mutual esteem and friendship as now exist not only between the scientists but also between the peoples of the British Commonwealth of Nations and the United States of America can spread to all peoples and to all countries.

In closing, Mr. President, I wish to express to the American Physical Society my gratitude for graciously providing this very pleasant setting for this occasion and thus lending indorsement to the honor conferred upon me by The Physical Society of London.

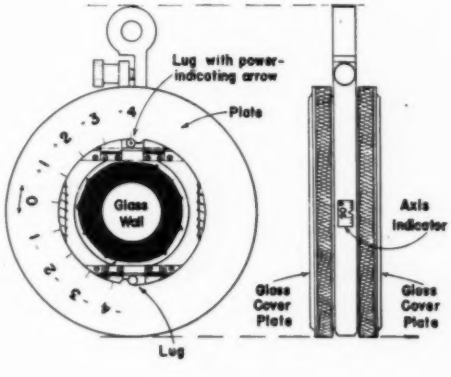
RESEARCH, after all, is nothing but prospecting for knowledge. Whenever someone sets out on an organized program to uncover new knowledge about a subject of which he knows very little, he is engaging in research. There is nothing mysterious about it, nothing highbrow. It is the most natural thing in the world.

CHARLES F. KETTERING in "Prospecting for Knowledge"
in the *Bakelite Review*, January, 1942

Résumé of Recent Research

A New Type of Variable Focus Lens*

The optical system of the eye is made up of two main elements: a fixed focus lens (the cornea) and an auxiliary lens of variable focus (the so-called "crystalline" lens). While great progress has been made in the design of fixed-focus lenses to aid defective vision, a practical lens of variable focus has not been made available. Attempts to produce lens variability have resulted either in complex optical systems, such as the "zooming focus" devices and the new oscillating lens systems used on cinema cameras, or in variable but optically unsatisfactory single lenses. The single lenses, for the most part, consist of liquid bounded by two transparent walls which are segments of spheres placed back to back. By constricting the periphery of these liquid-filled semi-spheres, or by increasing the pressure of the liquid within them, the attempt is made to maintain spherical surfaces while changing the radii thereof. These two things cannot simultaneously be done without actually altering the surface areas. In the absence of suitably elastic and transparent material such attempts result in distortion of the lens surfaces.



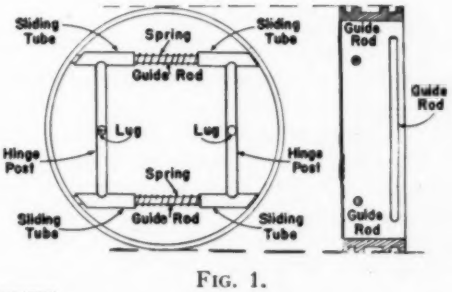


FIG. 1.

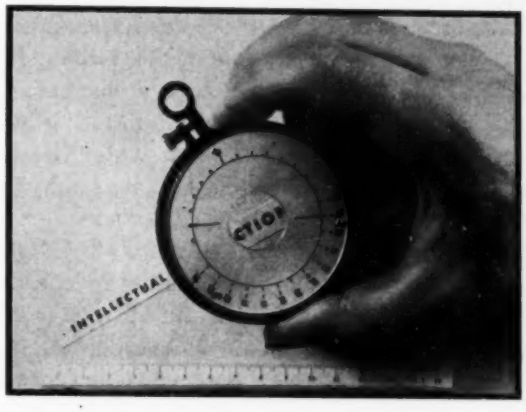


FIG. 2.

It is possible, however, to change the radius of a cylindrical segment without appreciably affecting its area and to produce a spherical refracting effect by means of two cylindrical segments, crossed at right angles. A lens embodying these two principles was designed and constructed at Ohio State University.[1]

The essential refracting unit consists of two plane disks of glass with clear liquid held between them by a rubber dam which connects the edges of the glass surfaces. For flexibility the optimum thickness of the glass was found to be about 0.07 mm. Clamped only by rubber and protected by glass cover plates, the two thin disks are not easily broken.

To select any dioptral power within the limits of the mechanism (±3.50 diopters in the first instrument), the circular plate of Fig. 1 is rotated until the desired figure is opposite a fixed indicator arrow. The reverse side of the device is identical to the side shown except that this second side is oriented at right angles to the first, so that the axes of the two variable cylinders will be crossed. By suitable rotation of the two circular plates with respect to each other, cylindrical, spherical, and sphero-cylindrical powers may be obtained.

This instrument (Fig. 2) was designed primarily for use in measuring errors of refraction of the eye to replace the ordinary method of

* Patent No. 2,269,905 (1942).

[1] Robert Graham, J. Opt. Soc. Am. **30**, 560–563 (1940).

measuring such errors by trying lens after lens before the eye until a combination is found which neutralizes them. In orthoptic or eye-training devices rotary prisms are frequently employed to alter gradually the amount of convergence of the two eyes while the accommodation remains constant. The variable lens now complements this practice by making it possible to induce gradual changes in the accommodation while the convergence remains constant. The lens may also find uses in certain types of cameras, where the proper focus may be procured merely by altering the focal length of the lens instead of by changing the separation between lens and film, as is done at present.

It also seems possible that variable lenses as described above could be made small enough so that, in certain cases, they could be worn before the eyes to replace bifocal lenses.

New Books

Photoelasticity, Volume I

By Max Mark Frocht. Pp. 411+xxvi, 23½×16 cm. John Wiley and Sons, Inc., New York, 1941. Price $6.00.

This is the first of two volumes on the subject of stress-optics. It covers the "essential, basic and practical" aspects of the subject and is intended for engineering reference and instruction. More advanced topics have been reserved for Volume II which is in preparation.

The photoelastic technique has progressed materially since Coker and Filon published their treatise. No other text has been available in English, although there is a voluminous but somewhat scattered literature on the subject. The present work therefore fills a need which has often been expressed but which has not been filled because of the labor involved in organizing what is essentially the first text to place stress-optics on an engineering basis.

The treatment includes a large number of illustrated examples of photoelastic analysis. This tends to clarify material which deals with basic principles not currently well understood by the engineering profession. The author points out that conventional analysis often fails to provide detailed stress distributions in machine and structural parts and that in many such cases the photoelastic procedure offers the only reasonable approach. Many photographs of stress patterns are included. These exhibit a technique and perfection for which the author is well known among his associates.

A significant lack in Volume I is an adequate treatment of the conditions of elastic similitude and it is hoped that this will be presented in some detail in Volume II. In other respects however, the reviewer wishes to indicate great admiration for the excellent manner in which the subject has been treated.

R. Weller
Naval Ordnance Laboratory

Acoustic Design Charts

By Frank Massa. Pp. 228+xiv, Charts 107, 23½×16 cm. The Blakiston Company, Philadelphia, 1942. Price $4.00.

This book, in a convenient size for reference purposes, contains 107 charts of useful data in the field of acoustical engineering. These are divided into ten sections according to subject matter. Each chart is explained by a short paragraph of text accompanying it, and its application is illustrated by a sample problem with solution. The book cannot claim to present any original material that cannot be found elsewhere in standard textbooks and in the literature. However, these charts are extremely handy for the person working in the acoustical field, who probably has drawn up similar charts in connection with his work —but not as many, nor as well designed and conveniently arranged as in this book. It is possible to think of additional charts that might be included, for instance directional radiation characteristics are given for a vibrating piston and a thin ring, but the well-known line array is omitted. The comment could also be made that no derivations of the charts are given and that charts based on experimental data are included without distinction from those representing exact mathematical relations.

These are very minor comments which do not detract from the value of the book as a whole, which should be a welcome addition to the library of all those interested in the design and construction of electro-acoustic apparatus.

E. Dietze
Division of National Defense
Columbia University

Symposium on New Methods for Particle Size Determination in the Subsieve Range

Pp. 111+v, Figs. 25, 15×23 cm. Published by the American Society for Testing Materials, Philadelphia, 1941. Price $1.25.

The papers and discussion presented at the Symposium on New Methods for Particle Size Determination in the Subsieve Range, held during the 1941 Spring Meeting of the American Society for Testing Materials, have recently been made available in book form. The titles of the papers and their authors are as follows: Methods for determining particle size distribution by Herbert E. Schweyer and Lincoln T. Work; Shape and surface of fine powders by the permeability method by P. C. Carman; Surface area of Portland cement by Paul S. Roller and P. V. Roundy, Jr.; An improved hydrometer method for use in fineness determinations by Alexander Klein; The determination of subsieve particle size distributions by sedimentation methods by S. W. Martin; The electron microscope in the

determination of particle size characteristics by James Hillier; A new method for measuring the surface areas of finely divided materials and for determining the size of particles by P. H. Emmett; Specific surface of pigments by adsorption from solution by Warren W. Ewing.

Copies may be obtained from A.S.T.M. Headquarters, 260 S. Broad Street, Philadelphia, Pennsylvania.

Here and There

Honors and Honored

Professor Albert Ball, Chairman of the Department of Physics at Cooper Union, who is retiring after 37 years of service, was given a testimonial dinner on May 28 by past and present members of the faculty of that school.

Professor Frank M. Simpson, Chairman of the Department of Physics at Bucknell University, retired on July 1 with the title Emeritus after 35 years of service in these capacities.

Dr. Richard C. Tolman, Professor of Physical Chemistry and Mathematical Physics at the California Institute of Technology, and Vice Chairman of the National Defense Research Committee, received the honorary degree of Doctor of Science at the commencement exercises of Prnceton University.

★

What Ho, the Attic!

Research workers seeking instruments required in their work but difficult to find are invited to communicate with D. H. Killeffer, Chairman of the newly appointed Committee on the Location of New and Rare Instruments of the National Research Council. The plan of the Committee's activity is to assist in locating needed instruments of types not ordinarily available through accustomed channels.

Assistance is particularly desired from owners and builders of instruments falling within the new or rare categories which might be made available to others through sale or for temporary use under mutually satisfactory conditions.

Communications should be addressed to D. H. Killeffer, *60 East 42nd Street, New York, New York.*

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New A.S.T.M. Officers

Recently elected officers of the American Society for Testing Materials for 1942–1943 are as follows: President (one-year term), Professor H. J. Ball, Lowell Textile Institute; Vice President (two-year term), Mr. P. H. Bates, National Bureau of Standards; Members of the Executive Committee (two-year term), Dr. R. P. Anderson, American Petroleum Institute; Dr. Maurice H. Bigelow, Plaskon Company, Inc.; Mr. J. H. Foote, The Commonwealth and Southern Corporation; Mr. Alexander Foster, Jr., Warner Company; Mr. Lawford H. Fry, Edgewater Steel Company.

Fellowships Available

Two fellowships are available in the Department of Geology, Massachusetts Institute of Technology, in the fields of Mineralogy and Structural Crystallography. One of these, a teaching fellowship, carries a stipend of $500 per academic year plus free tuition. Up to eighteen hours per week may be required of the holder of this fellowship. The duties consist of teaching and other assistance in connection with laboratory instruction in mineralogy and possibly optical crystallography. Preference will be given to a student specializing in mineralogy or crystallography.

Another fellowship, concerned chiefly with carrying out investigations in structural crystallography and allied fields, requires an investigator with a fairly good background in crystallography and preferably having had some experience in the x-ray study of crystal structures. This fellowship, which is of a more permanent nature, requires the full time of the incumbent and carries a stipend of about $2000.

Anyone interested in either of these fellowships should communicate with Professor M. J. Buerger, Department of Geology, Massachusetts Institute of Technology, Cambridge, Massachusetts, giving full details regarding their qualifications.

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Institute of Physics, London

During the past year the Institute of Physics in London has established an Industrial Radiology Group and an Electronics Group, the first such subject groups to be formed under the auspices of the Institute.

Recently elected officers who will assume duties on October 1, 1942 include: President, Sir Lawrence Bragg; Vice Presidents, Dr. W. Makower and Mr. T. Smith; Honorary Treasurer, Major C. E. S. Phillips; Honorary Secretary, Professor J. A. Crowther; Ordinary Members of the Board, Professor J. Chadwick, Professor J. D. Cockcroft, Mr. D. C. Gall, and Mr. E. B. Wedmore.

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Graduate Courses in Ultra-High Frequencies

The Graduate Electrical Engineering Department of the Polytechnic Institute of Brooklyn has again offered a group of five courses under the E.S.M.D.T. program for graduate students of physics or electrical engineering, as follows: Measurements at Ultra-High Frequencies; Cathode-Ray Oscillography; Generation and Reception of Ultra-High Frequencies; Ultra-High Frequency Antenna Systems and Reflectors; Ultra-High Frequency Wave Guides and Horns.

The program, which started the week of June 15, extends over eight weeks, each course meeting once a week for eight three-hour sessions and once a week for laboratory work of from five to eight sessions.

Contributed Original Research

See errata note: V. 14: 51–52

Theory of Vehicle Suspensions

PIERRE ERNEST MERCIER
Electro-Mechanical Research, Inc., Houston, Texas
(Received April 20, 1942)

The present theory of the suspension of vehicles is established on the assumption of perfect symmetry and of a linear dependence between elastic forces and the displacements of the wheels and deflections of the tires. The path of the vehicle rolling on uneven ground is supposed nearly straight, but no other important limitations are made in the fundamental hypothesis, so that the results cover any kind of suspension. The relative magnitude of the seven vibrations corresponding to bobbing, rolling, and pitching motions of the body, and to the four specific vibrations of the wheels, are discussed in view of their combined effects on the displacements of the body of the car. The general conclusion, which attempts to separate as much as possible the respective frequencies to avoid beats and also directly to improve comfort, is in accord with several practical experiments made in the industrial field.

INTRODUCTION

THE object of this paper is to present a theory of the suspension of vehicles, i.e., to analyze the motion of the vehicle body and wheels caused by translation over uneven ground. Conclusions are drawn from this analysis which are of considerable interest in practical suspension design. All such conclusions are in agreement with experimental evidence.

Passage over uneven ground sets up oscillations both of the wheels and of the sprung mass. The former occur at much higher frequencies, but nevertheless have an important effect on the latter; this may seem paradoxical at first, but becomes clear as the theory is developed, and is in agreement with experimental data.

The sprung mass in general has three degrees of freedom, corresponding to motions in vertical translation or "bounce," pitch, and roll, respectively. Particular emphasis is laid on the importance of the relationship between these modes.

ASSUMPTIONS AND BASIC DATA

Springing and Damping

A sprung vehicle consists of a body supported on wheels to which it is connected by elastic suspension members, the wheels being fitted with tires which are also capable of elastic deflection.

The elastic suspension members (which may be leaf or coil springs, torsion bars, or air cylinders) are generally supplemented by some form of shock absorber or "damper." These are not usually made so powerful as to suppress oscillations altogether, as this would have a detrimental effect on comfort.

Damping forces of reasonable magnitude have but little effect on the frequency of vibrations, and in this particular problem even the effect on amplitude is comparatively small. For the sake of mathematical simplicity damping forces will therefore be left out of account. This simplification is further justified by the fact that when elastic characteristics are rationally determined the required degree of damping is quite low. The general validity of the conclusions will not be affected by this simplification. All loads are assumed directly proportional to displacements.

The effective mass of each wheel is equal to the mass of the wheel and a fictitious mass which represents the inertia of the wheel carrying members; secondary reactions due to unbalance of the members are not taken into consideration. This assumption is quite evident for independent wheel springing suspension and admissible for orthodox suspension.

But we shall deal with the most general type of suspension of which the orthodox axle suspension and the independent wheel springing are but particular cases, as will appear presently.

Type of Suspension

The type of suspension is determined by the interaction between different wheels; i.e., the load in the suspension member corresponding to one wheel caused by the position and displacement of other wheels. In an independent suspension, such interaction is entirely absent if we suppose the spring mass non-affected by the motion of the wheels: that means that in an independent suspension there is no direct interaction. In the axle type of suspension there is an interaction between the two front wheels and also between the two rear wheels. In this case, the principal parts of such interaction are elastic interactions due to the ratio between the mutual distance of both springs (corresponding to one axle) and the track itself (mutual distance of the two wheels on the same axle) and the secondary reactions mentioned above are not very important.

One might, however, also conceive suspension giving interaction between the two wheels on the same side, or any other combination. The most general suspension will have interaction between all wheels; such suspensions can be realized in practice and more will be said about them later. For example, mechanical means can be developed in such case by connecting members working preferably in pure tension or by using torsional members. To reduce the weight and the dimensions of such systems the magnitude of the displacements communicated to the connecting members by the wheel motions are generally small in comparison to the wheel displacements themselves; so the specific differences between the several types of inertia effects involved in rolling, pitching, etc., are very small despite the stress applied to those connective members, and the assumption that secondary reactions are negligible is also true in this case if the fictitious mass added to the mass of each wheel is correctly adjusted.

Four-Wheel Vehicles

The treatment will be restricted to four-wheel vehicles. For the sake of simplicity, certain symmetry conditions are assumed, *viz*:

(a) Tracks of front and rear wheels are equal.
(b) All wheels and tires are identical.
(c) The vehicle has two vertical planes of symmetry (fore-and-aft and transverse).

The last assumption means that the c.g. is taken to be midway between front and rear wheels, and that front and rear suspension members are the same. Although this may seem an appreciable departure from reality (as far as practical vehicles are concerned), any more general assumption would lead to complications which would obscure the results without, however, modifying their meaning to any appreciable extent. Furthermore, the general method given here may be extended if desired to cover any special case.

DEVELOPMENT OF THEORY

Basic Displacements

Wheels are numbered in rotation round the vehicle: 1, 2, 3, 4, and corresponding suffixes will refer to the appropriate wheel, suffix "0" referring to the complete vehicle.

Figure 1 shows the relative positions and

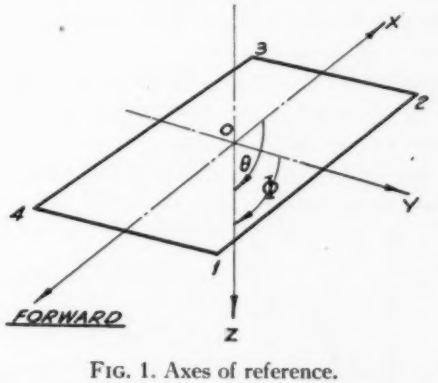

FIG. 1. Axes of reference.

displacements of the wheels and the axes of reference. Rotations corresponding to roll and pitch are denoted by Φ and θ, respectively. It will be noted that the positive direction of θ is the opposite of that usually taken (in order to maintain symmetry in the expressions for roll and pitch).

The plane of reference (XY plane) is the ground when even. Unevenness of the ground may be denoted by rises or "motions" under the wheels, the motions under the four wheels being denoted by z_1, z_2, z_3, and z_4, respectively, the mean rise being denoted by z_0.

Let us consider any one wheel, say wheel 1. (See Fig. 2.) When the ground under this wheel rises by an amount $-z_1$, this is taken up by compression of the tire, deformation of the suspension member, and rise of the sprung mass above the plane of reference. Let these be denoted by $-z_1'$, $-z_1''$, and $-z_1'''$, respectively, for the wheel 1 (similarly, with suffixes 2, 3, and 4 for the other wheels). We must then have

$$z_1'''=z_1+z_1'+z_1'', \tag{1}$$

and similarly for the other wheels (the quantity z_1'' is the deformation of the suspension member corresponding to wheel 1 as measured at the wheel center, i.e., the vertical movement of the wheel center relative to the sprung mass).

All quantities are measured from the static position as origin. Similarly, loads are measured from the static loads as origin, since static deflections and gravity loads have no effect on vibration phenomena of the type considered, which are only concerned with displacements from the static position. It should be noted that quantities z' and z'' are negative when the suspension members and tires *compress*, since z is positive downwards.

It will be useful in the following development of the theory to express motions of, or corresponding to, individual wheels in terms of net vertical translation ("bounce"), pitch, roll, and warping. Let us first consider motion of the ground, as defined by vertical motions of the wheel contact points. Referring to Fig. 3, let P_1, P_2, P_3, and P_4 be the contact points of the four wheels, which have risen by amounts z_1, z_2, z_3, and z_4 above the plane of reference (undistorted ground plane). Let N, N' be the midpoints of P_1P_4 and P_2P_3 and M, M' be the midpoints of P_1P_2 and P_3P_4, respectively. Roll is then defined as the slope of MM', pitch as the slope of NN', and warp as half the relative inclination between P_1P_4 and P_2P_3. We then have:

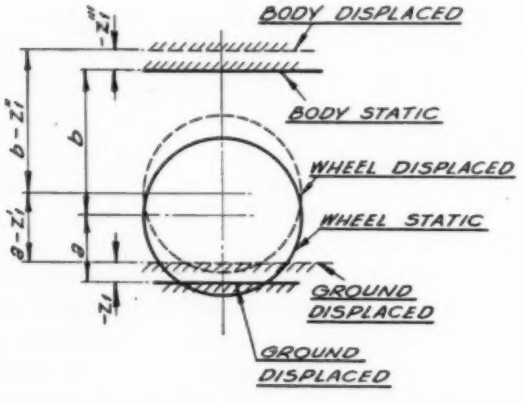

FIG. 2. Displacements at wheel "1."

Vertical translation of ground

$$z_0=(z_1+z_2+z_3+z_4)/4. \tag{2}$$

$$\text{Roll } \Phi_0=(z_1+z_2-z_3-z_4)/2T, \tag{3}$$

where T = track.

$$\text{Pitch } \theta_0=(z_1-z_2-z_3+z_4)/2W, \tag{4}$$

where W = wheelbase.

$$\text{Warp } \psi_0=(z_1-z_2+z_3-z_4)/2T, \tag{5}$$

ψ_0 being $\frac{1}{2}$ the warp angle. The roll, pitch, and warp angles are those of the distorted ground relative to the undistorted reference plane (angles being taken as equal to their tangents).

Similarly, if z_0', Φ_0', θ_0', ψ_0', are, respectively, the vertical translation, roll, pitch, and warp of the wheel center plane relative to the actual ground plane, these can be expressed in terms of tire deflections z_1', z_2', z_3', z_4' by equations of the same form as (2), (3), (4), and (5).

The net deformations of the suspension are similarly expressed in terms of z_1'', z_2'', z_3'', z_4'', these deformations being z_0'', Φ_0'', θ_0'', ψ_0'' for vertical translations, roll, pitch, and warp, respectively.

The net motions of the suspended mass, which are denoted by z_0''', Φ_0''', θ_0''', ψ_0''' for vertical translation, roll, pitch, and warp, respectively, are similarly expressed in terms of z_1''', z_2''', z_3''', z_4'''. The chassis is assumed to be perfectly rigid in torsion, and therefore $\psi_0'''=0$.

The relation between the motions of the ground, tires, suspension, and body have been expressed for each wheel in Eq. (1). Similar relationships apply to the net motions, and hence we have:

$$\begin{aligned} z_0'''&=z_0+z_1'+z_0'', \\ \Phi_0'''&=\Phi_0+\Phi_0'+\Phi_0'', \\ \theta_0'''&=\theta_0+\theta_0'+\theta_0'', \\ \psi_0'''&=\psi_0+\psi_0'+\psi_0''=0. \end{aligned} \tag{6}$$

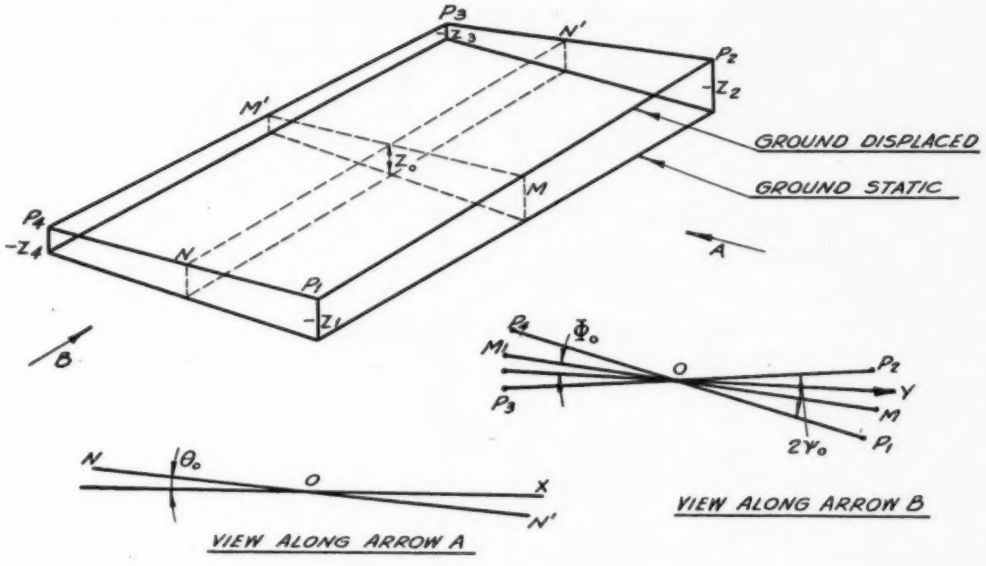

FIG. 3. Angular displacements.

Loads in Suspension Members

We have said that in the most general type of suspension the load in the suspension member corresponding to any one wheel is affected by the displacements of every other wheel. Hence we can write for the load on wheel 1:

$$F_1 = -(A_1 z_1'' + B_1 z_2'' + C_1 z_3'' + D_1 z_4'') \quad (7)$$

(loads being assumed proportional to deflections), where as previously stated z_1'', z_2'', z_3'', z_4'' measure the deflections of the four wheel centers relative to the sprung mass.

The above equation can be written

$$\begin{aligned} -F_1 = {} & F(z_1'' + z_2'' + z_3'' + z_4'') \\ & + R(z_1'' + z_2'' - z_3'' - z_4'') \\ & + P(z_1'' - z_2'' - z_3'' + z_4'') \\ & + H(z_1'' - z_2'' + z_3'' - z_4''), \end{aligned} \quad (8)$$

where

$$\begin{aligned} A_1 &= F + R + P + H, \\ B_1 &= F + R - P - H, \\ C_1 &= F - R - P + H, \\ D_1 &= F - R + P - H. \end{aligned} \quad (9)$$

Similar expressions, with the signs changed where relevant, apply to the other wheels. Thus for wheel 2, the coefficients in the expression corresponding to (7) will be A_2, B_2, C_2, D_2, where

$$A_2 = F + R - P - H, \quad \text{etc.}$$

(Equation (8) and corresponding expressions are always valid since the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$$

is not zero.)

Now the expressions in brackets in Eq. (8) are precisely the expressions given in the previous paragraph for the net deformation of the suspension in bounce, roll, pitch, and warp, respectively, and Eq. (8) can therefore be written:

$$\begin{aligned} -F_1 = {} & 4Fz_0'' + 2RT\Phi_0'' \\ & + 2PW\theta_0'' + 2HT\psi_0''. \end{aligned} \quad (10)$$

From a consideration of this expression the coefficients F, R, P, H acquire a precise physical significance, since it is apparent that they represent the rigidity of the whole suspension in bounce, roll, pitch, and warp, respectively. These four rigidities are written ξ, λ, μ, ν, where

$$\begin{aligned} -\xi &= -4F = dV/dz_0'', \\ -\lambda &= -4RT^2 = dL/d\Phi_0'', \\ -\mu &= -4PW^2 = dM_0/d\theta_0'', \\ -\nu &= -4HT = 2dN/d\psi_0'', \end{aligned} \quad (11)$$

where V, L, M_0, N are the net vertical load, rolling moment, pitching moment, and warping moment, respectively, acting on the whole suspension.

It may be explained for the sake of clarity that "warping" is a motion where the wheels, taken in rotation, go up, down, up, down. The warping of the ground has no direct effect on the suspended mass (which is assumed torsionally rigid) except insofar as it sets up torsional stresses. Nevertheless this "warping" has an important effect on the motions of the wheels, as will be shown later.

Equations of Motion for Wheels

The motion of the ground whether measured by z_0, ψ_0, Φ_0, θ_0, or by z_1, z_2, z_3, z_4 is a function of the shape of the ground surface and of the speed of translation u of the vehicle. The rear wheels are assumed to follow the same tracks as the front wheels, and hence:

$$\begin{aligned} z_2(t) &= z_1(t+W/u), \\ z_3(t) &= z_4(t+W/u), \end{aligned} \tag{12}$$

or

$$\begin{aligned} z_0(t+W/u)-z_0(t) &= -W/2(\theta_0(t)+\theta_0(t+W/u)), \\ \Phi_0(t+W/u)-\Phi_0(t) &= -1/2(\psi_0(t)+\psi_0(t+W/u)). \end{aligned} \tag{13}$$

It follows that we can at will consider one or the other of the series of parameters z_1, z_2, z_3, z_4 or z_0, ψ_0, Φ_0, θ_0, which may be considered as functions of time given in advance. The latter set will be chosen as it will lead to results which have a more direct physical interpretation.

The positions of the axes of roll and pitch and the c.g. of the vehicle have important effects if the motion is curvilinear or if the vehicle is accelerated. In the case of rectilinear motion (which we are considering), the height of the c.g. above the roll and pitch axes gives rise to secondary effects which are: (a) an anti-stabilizing force; (b) a side force (due to roll) and forward force (due to pitch). The first effect is very small and may be neglected, while the second is neutralized by the adherence of the tires (for roll) and the freedom in forward motion (for pitch), and hence may be also neglected.

The total motion of wheel 1 is given by:

$$z_0+z_0'+1/2(T(\Phi_0+\Phi_0'+\psi_0+\psi_0')+W(\theta_0+\theta_0')), \tag{14}$$

and similarly (with signs changed where necessary) for other wheels.

The suspension load on the same wheel is given by:

$$F_1=-\frac{1}{4}z_0''\xi-\frac{1}{8}\left(\frac{\lambda}{T}\Phi_0''+\frac{\mu}{W}\theta_0''+\frac{\nu}{T}\psi_0''\right). \tag{15}$$

Let K be the elastic rate of the tires. Hence the ground load on wheel 1 is

$$Kz_1'=K(z_0'+\tfrac{1}{2}\{T(\Phi_0'+\psi_0')+W\theta_0'\}. \tag{16}$$

If m is the mass of the wheel, we can now write the equation of motion of wheel 1 as follows:

$$\begin{aligned} &\frac{d^2}{dt^2}[z_0+z_0'+\tfrac{1}{2}\{T(\Phi_0+\Phi_0'+\psi_0+\psi_0')+W(\theta_0+\theta_0')\}] \\ &\qquad=\frac{1}{m}\left\{\frac{\xi}{4}z_0''+\frac{1}{8}\left(\frac{\lambda}{T}\Phi_0''+\frac{\mu}{W}\theta_0''+\frac{\nu}{T}\psi_0''\right)-K\left(z_0'+\frac{T}{2}(\Phi_0'+\psi_0')+\frac{W}{2}\theta_0'\right)\right\}. \end{aligned} \tag{17a}$$

Each wheel will have an equation of motion similar in form to (17a), but differing in the sign of the angular components. Let (17b), (17c), (17d), be the equations for wheels 2, 3, and 4. Then by adding all the four equations we shall eliminate all components but "bounce"; by adding (17a), (17b), $-$(17c), $-$(17d) we shall eliminate all components but roll, and similarly, *mutatis mutandis*, for pitch and warp. We then obtain equations of motion in the various modes for the wheel system

as a whole. These equations will be:

$$4\frac{d^2}{dt^2}[z_0+z_0']=\frac{1}{m}[\xi z_0''-4Kz_0'], \tag{18}$$

$$4\frac{d^2}{dt^2}[T(\Phi_0+\Phi_0')]=\frac{1}{m}\left[\frac{\lambda}{T}\Phi_0''-4KT\Phi_0'\right], \tag{19}$$

$$4\frac{d^2}{dt^2}[W(\theta_0+\theta_0')]=\frac{1}{m}\left[\frac{\mu}{W}\theta_0''-4KW\theta_0'\right], \tag{20}$$

$$4\frac{d^2}{dt^2}[T(\psi_0+\psi_0')]=\frac{1}{m}\left[\frac{\nu}{T}\psi_0''-4KT\psi_0'\right]. \tag{21}$$

Our final solution will require the partial elimination of the variables z_0, z_0', Φ_0, Φ_0', θ_0, θ_0', ψ_0, ψ_0'. This can be done by making use of Eq. (6), which it will be recalled relates the above variables (which represent ground motion and tire compression) to variables representing motion of the suspension and the suspended mass. We then get:

$$\frac{d^2}{dt^2}(z_0'''-z_0'')=\frac{1}{m}\left[\frac{\xi}{4}z_0''-Kz_0'''+K(z_0+z_0'')\right], \tag{22}$$

$$\frac{d^2}{dt^2}(\Phi_0'''-\Phi_0'')=\frac{1}{m}\left[\frac{\lambda}{4T^2}\Phi_0''-K\Phi_0'''+K(\Phi_0+\Phi_0'')\right], \tag{23}$$

$$\frac{d^2}{dt^2}(\theta_0'''-\theta_0'')=\frac{1}{m}\left[\frac{\mu}{4W^2}\theta_0''-K\theta_0'''+K(\theta_0+\theta_0'')\right], \tag{24}$$

$$-\frac{d^2\psi_0''}{dt^2}=\frac{1}{m}\left[\frac{\nu}{4W^2}\psi_0''+K(\psi_0+\psi_0'')\right]. \tag{25}$$

Equations of Motion for Complete System

Let M be the suspended mass and let r^2M and ρ^2M be its moments of inertia in roll and pitch, respectively, r and ρ being the radii of gyration about the respective axes. The equations of motion of the suspended mass will then be:

$$\frac{d^2z_0'''}{dt^2}=-\frac{1}{M}\xi z_0'', \tag{26}$$

$$\frac{d^2\Phi_0'''}{dt^2}=-\frac{1}{r^2M}\lambda\Phi_0'', \tag{27}$$

$$\frac{d^2\theta_0'''}{dt^2}=-\frac{1}{\rho^2M}\mu\theta_0''. \tag{28}$$

We may substitute for d^2z_0'''/dt^2 in Eq. (22) from Eq. (26), giving an equation which we shall call (22a). Equation (22a) and Eq. (26) form the "system" S_1:

$$S_1\left|\begin{array}{l}\dfrac{d^2z_0''}{dt^2}+\left[\xi\left(\dfrac{1}{M}+\dfrac{1}{4m}\right)+\dfrac{K}{m}\right]z_0''-\dfrac{K}{M}z_0'''=-\dfrac{K}{m}z_0, \qquad (22a)\\[2ex] \dfrac{d^2z_0'''}{dt^2}+\dfrac{\xi}{M}z_0''=0. \qquad (26)\end{array}\right.$$

This "system" completely defines the "bounce" (i.e., mean vertical translation) of the suspended mass and the suspension in terms of the mean vertical rise z_0 of the ground surface.

Similarly we can derive systems S_2, S_3, and S_4 which will define roll, pitch, and warp of the suspended mass and suspension (suspension only in the case of warp) in terms of the roll, pitch, and warp of the ground plane. Thus:

$$S_2 \left| \begin{aligned} &\frac{d^2\phi_0''}{dt^2}+\left[\lambda\left(\frac{1}{4mT^2}+\frac{1}{r^2M}\right)+\frac{K}{m}\right]\Phi_0''-\frac{K}{m}\Phi_0'''=-\frac{K}{m}\Phi_0. && (23a)\\ &\frac{d^2\Phi_0'''}{dt_2}+\frac{\lambda}{r^2M}\Phi_0''=0, && (27)\end{aligned}\right.$$

$$S_3 \left| \begin{aligned} &\frac{d^2\theta_0''}{dt^2}+\left[\mu\left(\frac{1}{4mW^2}+\frac{1}{\rho^2M}\right)+\frac{K}{M}\right]\theta_0''-\frac{K}{m}\theta_0'''=-\frac{K}{m}\theta_0, && (24a)\\ &\frac{d^2\theta_0'''}{dt^2}+\frac{\mu}{\rho^2M}\theta_0''=0, && (28)\end{aligned}\right.$$

$$S_4 \left| \frac{d^2\psi_0''}{dt^2}+\left[\frac{\nu}{4mW^2}+\frac{K}{m}\right]\psi_0''=-\frac{K}{m}\psi_0. \right. \qquad (25)$$

S_4 is merely a repetition of Eq. (25). With the exception of this last system, the above are all of the same form and may be completely solved if the independent variable (which represents "motion" of the ground under the vehicle) is known as a function of time.

STUDY OF MOTION

General Remarks

It will be shown below that the system which constitutes a four-wheel vehicle has seven natural modes of oscillation, each with its own frequency; two modes each correspond to bounce, roll, and pitch, and one to warp. The number of different modes can be deduced *a priori* from the fact that the system has seven degrees of freedom, *viz.*, the four compressions of the tires and the bounce, roll, and pitch of the body.

To analyze the motion we must solve the equation systems S_1 to S_4. The general solution is the sum of the general solution with the right-hand sides zero and a particular solution with particular values of the right-hand sides. The general solution will give us the natural frequencies of the system, and the complete solution will give the total motion when passing over a bump of known shape. A representative shape of ground bump will be selected for this purpose, a procedure which is justified by the fact that the actual shape chosen has no influence on the general conclusions to be drawn from the results of the analysis.

We shall begin by studying the system S_1, i.e., the motion in the "bounce" mode.

General Solution of Equations of Motion

The general solution of system S_1 (Eq. (24), and Eq. (22a) with R.H.S. = zero) will be of the type

$$pz_0''+qz_0'''=A\sin(\omega t+\epsilon),$$

and p and q are coefficients to be determined. Hence

$$p\frac{d^2z_0''}{dt^2}+q\frac{d^2z_0'''}{dt^2}=-\omega^2A(pz_0''+qz_0''').$$

Substituting this in system S_1 to eliminate d^2z_0''/dt^2 and d^2z_0'''/dt^2,

$$p\left[\xi\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}\right]z_0''-p\frac{K}{m}z_0'''+q\frac{\xi}{M}z_0''-\omega^2(pz_0''+qz_0''')=0.$$

From which we may equate coefficients of z_0'' and z_0''' to zero, getting:

$$p\left[\xi\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}\right]+q\frac{\xi}{M}-\omega^2p=0, \tag{26}$$

$$-p\frac{K}{m}-\omega^2q=0. \tag{27}$$

Hence we get

$$\begin{vmatrix} \xi\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}-\omega^2 & \frac{\xi}{M} \\ -\frac{K}{m} & -\omega^2 \end{vmatrix}=0, \tag{27a}$$

which gives us two roots for ω. Let these roots be α_1, β_1. From Eq. (27) we have

$$\frac{p}{q}\ -=\frac{\omega^2m}{K}.$$

The two values for the general solution of system S_1 are, therefore

$$z_0'''-\alpha_1{}^2\frac{m}{K}z_0''=I_1\cos(\alpha_1t+a_1), \tag{28}$$

$$z_0'''-\beta_1{}^2\frac{m}{K}z_0''=J_1\cos(\beta_1t+b_1). \tag{28a}$$

Particular Solution and Effect of Ground Bump

Let us consider the effect of one wheel passing over a bump of maximum height A_m, this maximum being reached at a distance x from the start of the bump, the shape being represented by an exponential law, *viz.*:

$$\frac{K}{m}z_0=\Gamma t\exp -rt, \tag{29}$$

where

$$r=\frac{1}{e}\cdot\frac{u}{x}\quad(u=\text{forward speed})$$

and

$$4\Gamma\frac{m}{K}=u\frac{A_m}{x},$$

z_0 being the mean rise of the ground, i.e., one-quarter of the height of the bump.

If we substitute this in the right-hand side of Eq. (22a), we shall be able to find the particular solution of the system S_1. It can be shown that the right-hand side of this solution will be the same

as for the general solution, and that the answer will be:

$$z_0''' - \frac{\alpha_1^2 m}{K} z_0'' = \frac{\Gamma}{\alpha_1^2 + r^2}(t+\tau_1') \exp -rt, \tag{30}$$

$$z_0''' - \frac{\beta_1^2 m}{K} z_0'' = \frac{\Gamma}{\beta_1^2 + r^2}(t+\tau_1'') \exp -rt, \tag{30a}$$

where

$$\tau_1' = \frac{2r}{\alpha_1^2 + r^2}, \quad \tau_1'' = \frac{2r}{\beta_1^2 + r^2}.$$

Motion of System

The complete solutions of the system S_1 are the sums of (28) and (30), and (28a) and (30a). Separating the variables z_0'' and z_0''', we get the motions of the suspended mass and the suspension members relative to the suspended mass. The motion of the suspended mass is given by:

$$z_0''' = \frac{1}{p_1'' - p_1'}\left[\{I_1 p_1' \cos(\alpha_1 t + a_1) - J_1 p_1'' \cos(\beta_1 t + b_1)\} + \exp -rt\left\{\frac{p_1'\Gamma}{\alpha_1^2 + r^2}(t+\tau_1') - \frac{p_1''\Gamma}{\beta_1^2 + r^2}(t+\tau_1'')\right\}\right] \tag{31}$$

and that of the wheels in relation to the mass by:

$$z_0'' = \frac{1}{p_1'' - p_1'}\left[\{J_1 \cos(\beta_1 t + b_1) - I_1 \cos(\alpha_1 t + a_1)\} + \exp -rt\left\{\frac{\Gamma}{\beta_1^2 + r^2}(t+\tau_1'') - \frac{\Gamma}{\alpha_1^2 + r^2}(t+\tau_1')\right\}\right] \tag{32}$$

where

$$p_1'' = \alpha_1^2 \frac{m}{K}, \quad p_1' = \beta_1^2 \frac{m}{K}.$$

The constants I_1, J_1, a_1, b_1, can be found from a knowledge of the initial conditions. If the system is assumed to be at rest at the instant of hitting the bump, then:

$$\tan a_1 = \frac{\alpha_1^2 - r^2}{2\alpha_1 r}, \quad I_1 = -\frac{\Gamma}{\alpha_1(\alpha_1^2 + r^2)}, \quad \text{etc.} \tag{33}$$

Motion in Roll, Pitch, and Warp

The motion in roll is determined in exactly similar fashion as that for bounce, by solving the equations of system S_2, which is of the same form as system S_1. The oscillations in roll have angular velocities α_2 and β_2 (angular velocity $= 2\pi \times$ frequency) which are the roots of the equation

$$\begin{vmatrix} \frac{\lambda}{4mT^2} + \frac{K}{m} + \frac{\lambda}{r^2 M} - \omega^2 & \frac{\lambda}{r^2 M} \\ -\frac{K}{m} & -\omega^2 \end{vmatrix} = 0.$$

The same ground bump which we have considered above (if occurring under wheel 1) is equivalent to a "rolling" of the ground plane given by

$$\frac{K}{m}\Phi_0 = \frac{2}{T}\Gamma t \exp -rt. \tag{34}$$

The complete solution of system S_2 will then give the rolling motions of the suspended mass (Φ_0''') and of the wheels in relation to the mass (Φ_0'') in a form identical with that of Eqs. (31) and (32). The suffix 2 will be used for the constants involved in roll.

The motion for pitch is obtained in similar fashion by solving equation system S_3, the natural angular velocities in this case being α_3 and β_3.

The solution of Eq. (25) or S_4 will give the wheel motion in warp (the suspended mass cannot warp), the general solution being:

$$\psi_0'' = -I_4 \cos(\alpha_4 t + a_4), \tag{35}$$

where

$$\alpha_4 = \left(\frac{\nu}{4mT^2} + \frac{K}{m}\right)^{\frac{1}{2}}.$$

Net Motion

We have now analyzed the seven distinct vibrations which are excited in the wheels and suspension by the passage of one wheel over a given obstacle. Except for the seventh (warp), these vibrations are all transmitted in various degrees to the suspended mass. The vibrations contain a sinusoidal component and a non-periodic component, the latter representing the shape of the obstacle, which is reproduced in the motions of the suspension and the sprung mass with some reduction of amplitude and change in phase.

These motions may be grouped as follows, for the suspended mass: the three principal vibrations correspond to angular velocities β_1, β_2, β_3, and to bounce, roll, and pitch, respectively. The three vibrations corresponding to angular velocities α_1, α_2, α_3 (and also to bounce, roll, and pitch), may be regarded as terms due to coupling of the suspended mass with the wheels. The only effect of the warping vibration (α_4) on the suspended mass is to induce torsional stresses in the chassis.

Similarly for the wheels the terms in α_1, α_2, α_3, α_4, may be regarded as the principal vibrations and the terms in β_1, β_2, β_3 as coupling effects with the suspended mass.

The net motion of wheel 1 (relative to the body) is given by:

$$z_1'' = z_0'' + \frac{T}{2}\Phi_0'' + \frac{W}{2}\theta_0'' + \frac{T}{2}\psi_0'', \tag{36}$$

i.e.,

$$\begin{aligned} z_1'' = {} & \frac{1}{p_1''-p_1'}\{J_1\cos(\beta_1 t+b_1) - I_1\cos(\alpha_1 t+a_1)\} + \frac{T}{2(p_2''-p_2')}\{J_2\cos(\beta_2 t+b_2) - I_2\cos(\alpha_2 t+a_2)\} \\ & + \frac{W}{2(p_3''-p_3')}\{J_3\cos(\beta_3 t+b_3) - I_3\cos(\alpha_3 t+a_3)\} - \frac{T}{2}I_4\cos(\alpha_4 t+a_4) \\ & + \frac{\Gamma\exp -rt}{2}\left[t\left\{\frac{1}{(\beta_1^2+r^2)(p_1''-p_1')} + \frac{1}{(\beta_2^2+r^2)(p_2''-p_2')} + \frac{1}{(\beta_3^2+r^2)(p_3''-p_3')}\right.\right. \\ & \left. - \frac{1}{(\alpha_1^2+r^2)(p_1''-p_1')} - \frac{1}{(\alpha_2^2+r^2)(p_2''-p_2')} - \frac{1}{(\alpha_3^2+r^2)(p_3''-p_3')} - \frac{1}{(\alpha_4^2+r^2)}\right\} \\ & + \frac{\tau_1''}{(\beta_1^2+r^2)(p_1''-p_1')} + \frac{\tau_2''}{(\beta_2^2+r^2)(p_2''-p_2')} + \frac{\tau_3''}{(\beta_3^2+r^2)(p_3''-p_3')} - \frac{\tau_1'}{(\alpha_1^2+r^2)(p_1''-p_1')} \\ & \left. - \frac{\tau_2'}{(\alpha_2^2+r^2)(p_2''-p_2')} - \frac{\tau_3'}{(\alpha_3^2+r^2)(p_3''-p_3')} - \frac{\tau_4'}{\alpha_4^2+r^2}\right]. \end{aligned} \tag{37}$$

(Note: Coefficients I_1 and J_1 have one dimension in length. Coefficients I_2, J_2, etc., are non-dimensional.)

It should be noted that the elimination of the parameter ν_1, the rigidity in warp (i.e., making this equal zero) will not modify the form of the preceding results; its only effect would be to eliminate torsional stresses on the chassis and to reduce α_4.

Integration of the equations of motion can also be performed for variations of ground shape of sinusoidal form, or form which may be expressed in terms of sinusoidal functions.

DISCUSSION OF RESULTS

Orthodox Suspensions

In the case of orthodox suspensions the relative rigidities of the suspension are interrelated by laws which are affected only by the proportions of the chassis (to a small extent). This means that there is a relation between the four coefficients F, R, P, H or the four quantities A_1, B_1, C_1, D_1 and their homologues.

For example in the case of an orthodox suspension with axles there is no interaction between wheels on the same side of the vehicle, and therefore:

$$B_1=0,\quad A_2=0,\quad A_3=0,\quad B_4=0,\quad C_1=0,\quad D_2=0,\quad D_3=0,\quad C_4=0,$$

which gives $F+R-P-H=0$, and $F-R-P+H=0$, i.e., $R=H$ and $F=P$, whence $\lambda=\nu$ and $\xi=\mu/W^2$.

We shall show that for suspensions of this type the only coupling between wheels which are not on the same side is through the suspended mass. Considering the bounce mode, the angular velocities α_1 and β_1 in this mode are given by Eq. (27a). If now we make M (the suspended mass) infinite in this equation, the root α_1 becomes $\alpha_1'=(\xi/4m+K/m)^{\frac{1}{2}}$. Similarly for pitch, the root α_3 becomes $\alpha_3'=(\mu/mW^2+K/m)^{\frac{1}{2}}$. Hence α_1 and α_3 are equal, but the signs of the vibrations corresponding to α_3 will be opposite for wheels on opposite axes. Thus if one of the front wheels passes over a bump the bounce and pitch oscillations in the rear wheels will be of the same magnitude but opposite sign (i.e., out of phase by 180 degrees), and the net oscillation in the rear wheels due to pitch and bounce will be zero. Hence the only coupling must be through the suspended mass, since, when this mass is made infinite, the coupling disappears.

In the case of a symmetric car with four independent wheels, the only coupling between any wheels is through the intermediary of the suspended mass. In this case $F=P=H=R$.

Causes of Discomfort in Orthodox Suspensions

Due to coupling effects the seven frequencies of vibration are always distinct. However, the angular velocities α_1, α_2, α_3, α_4, are in fact very close in value, and this is the reason for the "beat" phenomena which are the main cause of discomfort in cars with orthodox suspensions if there is inadequate damping.

"Beats" occur if there are two superimposed oscillations, the frequencies of which are fairly close. We shall now proceed to study this in more detail, and to determine the degree of closeness between some of the frequencies and the factors which influence it, for the case of an orthodox independent wheel suspension.

In this case we have $F=R$, and therefore $\lambda=T^2\xi$.

The factors $1/M+1/4m$ and $1/Mr^2+1/4mT^2$ occur in the expressions for bounce and roll frequencies. We may write the latter expression as

$$\frac{1}{T^2}\left(\frac{T^2}{r^2}\cdot\frac{1}{M}+\frac{1}{4m}\right).$$

The radius of gyration in roll r being normally less than the track T, we may write

$$\frac{T^2}{r^2}\cdot\frac{1}{M}+\frac{1}{4m}=u\left[\frac{1}{M}+\frac{1}{4m}\right],$$

where "u" is greater than unity.

Now the vibration frequencies in bounce and roll are given by:

$$\begin{vmatrix} \xi\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}-\omega^2, & \frac{\xi}{m} \\ -\frac{K}{m}, & -\omega^2 \end{vmatrix}=0$$

and

$$\begin{vmatrix} \frac{\lambda}{T^2}u\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}-\omega^2, & \frac{\lambda}{r^2M} \\ -\frac{K}{m}, & -\omega^2 \end{vmatrix}=0,$$

the last of which (roll) may be written: (since $\lambda/T^2=\xi$)

$$\begin{vmatrix} \xi u\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}-\omega^2, & \frac{u'\xi}{M} \\ -\frac{K}{m}, & -\omega^2 \end{vmatrix}=0, \quad (38)$$

where

$$u'=\frac{T^2}{r^2}, \quad (u'>1),$$

i.e.,

$$\omega^4-\omega^2\left[\xi u\left(\frac{1}{M}+\frac{1}{4m}\right)+\frac{K}{m}\right]+u'\frac{\xi K}{Mm}=0. \quad (39)$$

This equation will give the roll frequencies by inserting the proper values for u and u', and also the bounce frequencies by making u and u' equal to unity.

Increasing the rigidity in roll independently of that in bounce is equivalent to increasing u and u' in Eq. (39), i.e., to increasing both the sum and product of the frequencies in roll as compared with bounce. The effect therefore would be to "decouple" the bounce and roll oscillations and to reduce beats.

In an axle type of suspension $\lambda<\xi/T$ and hence u and u' tend to approach unity. Hence the frequencies for roll and bounce are very close together. This is one of the main reasons for the advantage of independent suspensions over those of the axle type.

"Interaction" Suspensions

Our study has covered the most general type of suspension of which the axle type and the independent wheel type are but particular instances. It is apparent from what has been said above that the independent wheel suspension has no fundamental advantages from a general standpoint, although it may be preferable to the axle type.

The fundamental solution is a suspension in which the four quantities F, R, P, H, or in other words the rigidities in bounce, roll, pitch, and warp, can be selected entirely independently of each other, thus allowing the choice of the best compromise between stability and comfort (a step in that direction is the introduction of roll stabilizers). In particular, it is possible to eliminate rigidity in warping and thus suppress torsion loads on the chassis.

Such suspensions have actually been devised and tested by the author. They have proved to be very much superior to any other type, both from the point of view of comfort and road holding; in fact, it has been possible in some cases to dispense with shock absorbers without detrimental effect on comfort.

Calendar of Meetings

September

7– 9 Mathematical Association of America, Ithaca, New York
7–11 American Chemical Society, Buffalo, New York
8–11 American Mathematical Society, Ithaca, New York
9–11 American Institute of Electrical Engineers, Vancouver, B. C.
29–Oct. 1 Technical Association of Pulp and Paper Industry, Boston, Massachusetts

October

5– 9 National Safety Council, Chicago, Illinois
12–14 American Society of Mechanical Engineers, Rochester, New York
12–16 American Welding Society, Detroit, Michigan
12–16 American Society for Metals, Detroit, Michigan
19–24 American Society for Metals, Chicago, Illinois
19–24 American Society for Metals, Cleveland, Ohio
30–31 Optical Society of America, New York, New York
31 American Mathematical Society, New York, New York

November

7 American Physical Society, New York State Section, Albany, New York
16–18 American Institute of Chemical Engineers, Cincinnati, Ohio
27–28 American Mathematical Society, Notre Dame, Indiana
30–Dec. 4 American Society of Mechanical Engineers, New York, New York

December

27–Jan. 1 American Society of Naturalists, New York, New York
28–Jan. 2 American Association for the Advancement of Science, New York, New York
28–Jan. 2 American Standards Association, New York, New York

Calculation of Axially Symmetric Fields*

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A practical method is developed for calculating the distribution of scalar fields in axially symmetric systems. To obtain a complete solution a knowledge of the potential distribution along a cylinder of constant radius is required; in many cases the distribution along the cylinder can be estimated with sufficient accuracy by an inspection of the electrode configuration. The solution is first obtained in the form of an integral. This is evaluated for the class of boundary potential functions of the form

$$V_n(1, z) = \begin{cases} z^n, & z>0 \\ 0, & z<0. \end{cases}$$

In this expression, $V_n(1, z)$ is the potential along a cylinder of unit radius, z is measured along the axis of the cylinder, and n is a positive integer (zero included). Numerical values are given in tabular form for the cases $n=0$, 1, 2 and 3. It is then shown how the solution for the above boundary potential functions can be used to obtain the solution for an arbitrary distribution of potential along a cylinder of constant radius. In particular, the potential distribution of a simple electron lens consisting of two coaxial cylindrical electrodes of equal diameters is expressed in terms of the tabulated functions. The calculated equipotential lines for one such case are compared graphically with a plot made in an electrolytic plotting tank. The comparison between the calculated and experimental results is quite favorable.

THE increasing number of devices in which scalar fields with symmetry about a central axis occurs makes it desirable to have a simple method of calculating the field in such cases. In particular, such devices as kinescopes, image converters, etc., require a knowledge of the potential fields before a calculation of the electron trajectories can be made. In the past electrolytic plotting tanks have been used to obtain equipotential maps for particular electrode systems and a graphical method is then used to obtain the electron trajectories. The accuracy of such calculations is limited by the refinement of the apparatus used to obtain the potential plot. This paper gives a practical method for calculating the field in such cases.

Scalar potential fields must satisfy Laplace's equation; for axially symmetric fields it is convenient to consider this equation expressed in cylindrical coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \qquad (1)$$

where $V = V(r, z)$ is the potential at a radius r from the axis of symmetry and at a distance z from a reference plane perpendicular to the axis. Several general solutions of this equation have been given;[1] the particular solution used here is the Fourier integral

$$V(r, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k) I_0(kr) e^{ikz} dk, \qquad (2)$$

where $I_0(kr) = J_0(ikr)$ is the Bessel function of order zero with imaginary argument and where $a(k)$ is a function of the arbitrary parameter k; $a(k)$ must be chosen so that $V(r, z)$ satisfies the boundary conditions.

If the potential at some radius R is known as a function of z then $a(k)$ can be found by means of the Fourier transform

$$a(k) I_0(kR) = \int_{-\infty}^{\infty} V(R, z) e^{-ikz} dz \qquad (3)$$

or, if for convenience the radius at which the potential is known is taken as the unit of linear measure, then $R=1$ and (3) becomes

$$a(k) I_0(k) = \int_{-\infty}^{\infty} V(1, z) e^{-ikz} dz. \qquad (4)$$

From (4) $a(k)$ can be found explicitly in terms of the boundary potential $V(1, z)$. Substitution of $a(k)$ thus found in (2) yields

$$V(r, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(1, \zeta)\frac{I_0(kr)}{I_0(k)} e^{ik(z-\zeta)} dk d\zeta, \qquad (5)$$

* Taken from a thesis of the same title presented in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering (1941) at The Ohio State University.

[1] See, for example: L. M. Myers, *Electron Optics, Theoretical and Practical* (D. Van Nostrand Company, New York, New York 1939), pp. 87–91.

where the variable of integration in (4) is replaced by ζ to avoid confusion. Since the order of the integration in (5) is immaterial,[2] the potential can be expressed

$$V(r, z)=\int_{-\infty}^{\infty} V(1, \zeta)\, U(r, \xi)\, d\zeta, \tag{6}$$

where

$$U(r, \xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{I_0(kr)}{I_0(k)}e^{ik\xi}dk \tag{7}$$

and $\xi=z-\zeta$.[3]

Since (7) is independent of the boundary potential $V(1, z)$ it can be evaluated and the result used in (6) to obtain the potential distribution for the general case. To evaluate (7) the fraction $I_0(kr)/I_0(k)$ is first expanded into a series of partial fractions,[4] thus

$$\frac{I_0(kr)}{I_0(k)}=\sum_{\mu}\frac{I_0(i\mu r)}{(k-i\mu)I_0'(i\mu)}=\sum_{\mu}\frac{J_0(\mu r)}{iJ_1(\mu)(k-i\mu)}, \tag{8}$$

where the summation is taken over all roots μ of the equation $J_0(\mu)=0$. Then

$$U(r, \xi)=\frac{1}{2\pi i}\sum_{\mu}\int_{-\infty}^{\infty}\frac{J_0(\mu r)e^{ik\xi}}{J_1(\mu)(k-i\mu)}dk. \tag{9}$$

Evaluating (9) by a contour integration there results:

A. for positive μ

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ik\xi}}{k-i\mu}dk=\begin{cases}e^{-\mu\xi}, & \xi>0\\ 0, & \xi<0\end{cases} \tag{10}$$

B. for negative μ

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ik\xi}}{k-i\mu}dk=\begin{cases}0, & \xi>0\\ e^{\mu\xi}, & \xi<0.\end{cases} \tag{11}$$

Since there is a negative root numerically equal to every positive root of $J_0(x)=0$ and vice versa (9) can finally be written

$$U(r, \xi)=\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}e^{-\mu|\xi|}. \tag{12}$$

In terms of the series (12) Eq. (6) becomes

$$V(r, z)=\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}\int_{-\infty}^{\infty}V(1, \zeta)e^{-\mu|\xi|}d\zeta. \tag{13}$$

It is now expedient to apply (13) to the simple class of boundary functions of the form

$$V_n(1, z)=\begin{cases}0, & z<0\\ z^n, & z>0,\end{cases} \tag{14}$$

where n is a positive integer or zero. This yields

$$V_n(r, z)=\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}\int_{0}^{\infty}\zeta^n e^{-\mu|z-\zeta|}d\zeta. \tag{15}$$

For $z>0$ this becomes

$$\begin{aligned}\underset{z>0}{V_n(r, z)}=&\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}\left\{\int_0^z\zeta^n e^{-\mu(z-\zeta)}d\zeta\right.\\ &\left.+\int_z^{\infty}\zeta^n e^{-\mu(\zeta-z)}d\zeta\right\}\\ =&\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}\left\{e^{-\mu z}\left[e^{\mu\zeta}\left(\frac{\zeta^n}{\mu}-\frac{n\zeta^{n-1}}{\mu^2}\right.\right.\right.\\ &\left.\left.\left.+n(n-1)\frac{\zeta^{n-2}}{\mu^3}+\cdots\right)\right]_0^z\right.\\ &\left.+e^{\mu z}\left[e^{-\mu\zeta}\left(\frac{-\zeta^n}{\mu}-\frac{n\zeta^{n-1}}{\mu^2}\right.\right.\right.\\ &\left.\left.\left.-n(n-1)\frac{\zeta^{n-2}}{\mu^3}+\cdots\right)\right]_z^{\infty}\right\}.\end{aligned} \tag{16}$$

---

[2] The order of the integration in (5) is immaterial if the integral (4) converges. This is not true for boundary potentials $V(1, z)$ as defined in (14). The condition is satisfied, however, for functions of the form

$$V_n^*(1, z)=\begin{cases}0, & z>A\\ z^n, & 0<z<A\\ 0, & z<0\end{cases}.$$

This would require that the upper limit of integration in (15) be replaced by A. Since (15) converges uniformly as $A\to\infty$ the results here obtained by changing the order of integration in (5) are valid.

[3] For the calculation of paraxial electron trajectories the axial potential distribution is sufficient. A good approximation for this purpose is

$$U(0, \xi)\approx\frac{\omega}{2}\operatorname{sech}^2\omega\xi,$$

where $\omega=1.32$ giving

$$V(0, z)\approx\frac{\omega}{2}\int_{-\infty}^{\infty}V(1, \zeta)\operatorname{sech}^2\omega\xi d\zeta.$$

For a discussion of this equation see S. Bertram, "Determination of the axial potential distribution in axially symmetric fields," Proc. I.R.E. **28**, 418–420 (1940).

[4] B. O. Peirce, *A Short Table of Integrals* (Ginn and Company, 1929), pp. 13–14.

TABLE I.

	$z>0$	$z<0$
$V_0(r,z)$	$1-\sum_{+\mu}\frac{J_0(\mu r)}{\mu J_1(\mu)}e^{-\mu z}$	$\sum_{+\mu}\frac{J_0(\mu r)}{\mu J_1(\mu)}e^{\mu z}$
$V_1(r,z)$	$z+\sum_{+\mu}\frac{J_0(\mu r)}{\mu^2 J_1(\mu)}e^{-\mu z}$	$\sum_{+\mu}\frac{J_0(\mu r)}{\mu^2 J_1(\mu)}e^{\mu z}$
$V_2(r,z)$	$z^2+\sum_{+\mu}\frac{2J_0(\mu r)}{\mu^3 J_1(\mu)}[2-e^{-\mu z}]$	$\sum_{+\mu}\frac{2J_0(\mu r)}{\mu^3 J_1(\mu)}e^{\mu z}$
$V_3(r,z)$	$z^3+\sum_{+\mu}\frac{6J_0(\mu r)}{\mu^3 J_1(\mu)}\left[2z+\frac{e^{-\mu z}}{\mu}\right]$	$\sum_{+\mu}\frac{6J_0(\mu r)}{\mu^4 J_1(\mu)}e^{\mu z}$

One obtains from $V_0(r,z)$ at $z=0$ the relation

$$\sum_{+\mu}\frac{2J_0(\mu r)}{\mu J_1(\mu)}=1;$$

using this (16) reduces to

$$V_n(r,z)_{z>0}=z^n+\sum_{+\mu}\frac{2J_0(\mu r)}{J_1(\mu)}\left\{n(n-1)\frac{z^{n-2}}{\mu^3}+n(n-1)(n-2)(n-3)\frac{z^{n-4}}{\mu^5}+\cdots+\frac{n!e^{-\mu z}}{2(-\mu)^{n+1}}\right\}. \quad (17)$$

For $z<0$

$$V_n(r,z)_{z<0}=\sum_{+\mu}\frac{J_0(\mu r)}{J_1(\mu)}e^{\mu z}\int_0^\infty \zeta^n e^{-\mu\zeta}d\zeta=\sum_{+\mu}\frac{J_0(\mu r)n!e^{\mu z}}{J_1(\mu)\mu^{n+1}}. \quad (18)$$

Table I lists the first four of these functions ($n=0$, 1, 2, 3). The series are all convergent; the more important values are tabulated in Tables II–V.

TABLE II. $\sum_{+\mu}\frac{J_0(\mu r)}{\mu J_1(\mu)}e^{-\mu z}$

$z\backslash r$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.05	0.4669	0.4666	0.4656	0.4639	0.4610	0.4566	0.4495	0.4372	0.4126	0.3441
0.10	0.4341	0.4335	0.4316	0.4281	0.4226	0.4140	0.4005	0.3778	0.3347	0.2360
0.15	0.4019	0.4010	0.3983	0.3933	0.3854	0.3732	0.3545	0.3240	0.2706	0.1697
0.20	0.3706	0.3695	0.3660	0.3596	0.3497	0.3348	0.3121	0.2769	0.2201	0.1279
0.25	0.3404	0.3391	0.3349	0.3276	0.3161	0.2991	0.2740	0.2367	0.1808	0.1000
0.30	0.3116	0.3101	0.3054	0.2973	0.2847	0.2664	0.2401	0.2027	0.1501	0.0803
0.35	0.2841	0.2825	0.2776	0.2689	0.2557	0.2367	0.2103	0.1740	0.1259	0.0658
0.40	0.2583	0.2567	0.2515	0.2425	0.2290	0.2100	0.1842	0.1499	0.1065	0.0548
0.45	0.2342	0.2325	0.2273	0.2182	0.2047	0.1861	0.1614	0.1296	0.0907	0.0461
0.50	0.2118	0.2101	0.2049	0.1959	0.1827	0.1648	0.1416	0.1124	0.0778	0.0392
0.55	0.1910	0.1894	0.1843	0.1755	0.1629	0.1459	0.1243	0.0978	0.0671	0.0336
0.60	0.1719	0.1703	0.1654	0.1571	0.1451	0.1292	0.1092	0.0853	0.0581	0.0289
0.65	0.1545	0.1529	0.1482	0.1404	0.1291	0.1143	0.0961	0.0746	0.0504	0.0250
0.70	0.1385	0.1371	0.1327	0.1253	0.1148	0.1012	0.0846	0.0653	0.0440	0.0217
0.75	0.1241	0.1227	0.1186	0.1117	0.1020	0.0896	0.0745	0.0573	0.0384	0.0189
0.80	0.1110	0.1097	0.1059	0.0995	0.0906	0.0793	0.0657	0.0503	0.0337	0.0165
0.85	0.0991	0.0979	0.0944	0.0886	0.0805	0.0702	0.0580	0.0443	0.0295	0.0145
0.90	0.0885	0.0874	0.0841	0.0788	0.0714	0.0622	0.0512	0.0390	0.0260	0.0127
0.95	0.0789	0.0779	0.0749	0.0701	0.0634	0.0551	0.0453	0.0344	0.0228	0.0112
1.00	0.0703	0.0694	0.0667	0.0623	0.0563	0.0488	0.0400	0.0303	0.0201	0.0098
1.05	0.0626	0.0618	0.0593	0.0554	0.0499	0.0432	0.0354	0.0268	0.0177	0.0087
1.10	0.0557	0.0550	0.0527	0.0492	0.0443	0.0383	0.0313	0.0237	0.0157	0.0076
1.15	0.0495	0.0489	0.0469	0.0437	0.0393	0.0339	0.0277	0.0209	0.0138	0.0067
1.20	0.0440	0.0434	0.0417	0.0388	0.0349	0.0301	0.0245	0.0185	0.0122	0.0059
1.25	0.0391	0.0386	0.0370	0.0344	0.0309	0.0266	0.0217	0.0164	0.0108	0.0053
1.30	0.0347	0.0343	0.0329	0.0306	0.0274	0.0236	0.0192	0.0145	0.0096	0.0047
1.35	0.0309	0.0304	0.0292	0.0271	0.0243	0.0209	0.0170	0.0128	0.0084	0.0041
1.40	0.0274	0.0270	0.0259	0.0241	0.0216	0.0186	0.0151	0.0113	0.0075	0.0036
1.45	0.0243	0.0240	0.0230	0.0213	0.0191	0.0164	0.0134	0.0101	0.0066	0.0032
1.50	0.0216	0.0213	0.0204	0.0189	0.0170	0.0146	0.0119	0.0089	0.0059	0.0029
1.55	0.0192	0.0189	0.0181	0.0168	0.0150	0.0129	0.0105	0.0079	0.0052	0.0025
1.60	0.0170	0.0168	0.0160	0.0149	0.0133	0.0115	0.0093	0.0070	0.0046	0.0022
1.65	0.0151	0.0149	0.0142	0.0132	0.0118	0.0101	0.0082	0.0062	0.0041	0.0020
1.70	0.0134	0.0132	0.0126	0.0117	0.0105	0.0090	0.0073	0.0055	0.0036	0.0017
1.75	0.0119	0.0117	0.0112	0.0104	0.0093	0.0080	0.0065	0.0049	0.0032	0.0015
>1.75	0.800α	0.788α	0.754α	0.700α	0.625α	0.537α	0.434α	0.326α	0.214α	0.104α

$\alpha=e^{-2.4z}$

TABLE III. $\sum_{+\mu} \frac{J_0(\mu r)}{\mu^2 J_1(\mu)} e^{-\mu z}$

z\r	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.2674	0.2658	0.2607	0.2522	0.2398	0.2232	0.2016	0.1739	0.1377	0.0883
0.05	0.2432	0.2416	0.2366	0.2281	0.2158	0.1993	0.1779	0.1505	0.1150	0.0674
0.10	0.2207	0.2191	0.2141	0.2058	0.1937	0.1775	0.1567	0.1301	0.0964	0.0531
0.15	0.1998	0.1982	0.1934	0.1852	0.1735	0.1579	0.1378	0.1126	0.0813	0.0431
0.20	0.1805	0.1790	0.1743	0.1664	0.1551	0.1402	0.1212	0.0976	0.0691	0.0358
0.25	0.1627	0.1613	0.1568	0.1492	0.1385	0.1243	0.1065	0.0848	0.0591	0.0301
0.30	0.1465	0.1450	0.1408	0.1336	0.1235	0.1102	0.0937	0.0738	0.0509	0.0256
0.35	0.1316	0.1302	0.1262	0.1195	0.1100	0.0976	0.0824	0.0644	0.0440	0.0220
0.40	0.1180	0.1168	0.1130	0.1067	0.0979	0.0865	0.0726	0.0564	0.0382	0.0190
0.45	0.1057	0.1045	0.1011	0.0952	0.0870	0.0766	0.0640	0.0494	0.0333	0.0165
0.50	0.0946	0.0935	0.0903	0.0849	0.0774	0.0678	0.0564	0.0433	0.0291	0.0143
0.55	0.0845	0.0835	0.0805	0.0756	0.0687	0.0601	0.0498	0.0381	0.0255	0.0125
0.60	0.0754	0.0745	0.0718	0.0673	0.0611	0.0532	0.0439	0.0335	0.0224	0.0110
0.65	0.0673	0.0664	0.0640	0.0598	0.0542	0.0471	0.0388	0.0295	0.0196	0.0096
0.70	0.0600	0.0592	0.0569	0.0532	0.0481	0.0417	0.0343	0.0261	0.0173	0.0085
0.75	0.0534	0.0527	0.0507	0.0473	0.0427	0.0370	0.0303	0.0230	0.0152	0.0075
0.80	0.0475	0.0469	0.0451	0.0420	0.0379	0.0328	0.0268	0.0203	0.0134	0.0066
0.85	0.0423	0.0417	0.0401	0.0373	0.0336	0.0290	0.0237	0.0179	0.0119	0.0058
0.90	0.0376	0.0371	0.0356	0.0332	0.0298	0.0257	0.0210	0.0159	0.0105	0.0051
0.95	0.0334	0.0330	0.0316	0.0294	0.0265	0.0228	0.0186	0.0140	0.0093	0.0045
1.00	0.0297	0.0293	0.0281	0.0261	0.0235	0.0202	0.0165	0.0124	0.0082	0.0040
1.05	0.0264	0.0260	0.0249	0.0232	0.0208	0.0179	0.0146	0.0110	0.0072	0.0035
1.10	0.0234	0.0231	0.0221	0.0206	0.0185	0.0159	0.0129	0.0097	0.0064	0.0031
1.15	0.0208	0.0205	0.0196	0.0183	0.0164	0.0141	0.0115	0.0086	0.0057	0.0028
1.20	0.0185	0.0182	0.0174	0.0162	0.0145	0.0125	0.0101	0.0076	0.0050	0.0024
1.25	0.0164	0.0162	0.0155	0.0144	0.0129	0.0111	0.0090	0.0068	0.0044	0.0022
1.30	0.0145	0.0143	0.0137	0.0127	0.0114	0.0098	0.0080	0.0060	0.0039	0.0019
1.35	0.0129	0.0127	0.0122	0.0113	0.0101	0.0087	0.0071	0.0053	0.0035	0.0017
1.40	0.0114	0.0113	0.0108	0.0100	0.0090	0.0077	0.0063	0.0047	0.0031	0.0015
1.45	0.0102	0.0100	0.0096	0.0089	0.0080	0.0068	0.0055	0.0042	0.0027	0.0013
1.50	0.0090	0.0089	0.0085	0.0079	0.0071	0.0060	0.0049	0.0037	0.0024	0.0012
>1.50	0.332α	0.327α	0.313α	0.291α	0.260α	0.223α	0.180α	0.135α	0.089α	0.043α
					$\alpha = e^{-2.4z}$					

TABLE IV. $\sum_{+\mu} \frac{J_0(\mu r)}{\mu^3 J_1(\mu)} e^{-\mu z}$

z\r	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.1250	0.1237	0.1200	0.1137	0.1050	0.0937	0.0800	0.0637	0.0450	0.0237
0.05	0.1122	0.1111	0.1076	0.1019	0.0936	0.0832	0.0705	0.0556	0.0387	0.0199
0.10	0.1006	0.0996	0.0963	0.0910	0.0834	0.0738	0.0622	0.0487	0.0334	0.0169
0.15	0.0901	0.0891	0.0861	0.0812	0.0742	0.0654	0.0548	0.0426	0.0290	0.0145
0.20	0.0806	0.0797	0.0769	0.0724	0.0660	0.0580	0.0484	0.0373	0.0252	0.0125
0.25	0.0721	0.0712	0.0687	0.0645	0.0587	0.0514	0.0427	0.0328	0.0221	0.0109
0.30	0.0644	0.0636	0.0613	0.0574	0.0521	0.0455	0.0377	0.0288	0.0193	0.0095
0.35	0.0574	0.0567	0.0546	0.0511	0.0463	0.0403	0.0333	0.0254	0.0169	0.0083
0.40	0.0512	0.0505	0.0486	0.0454	0.0411	0.0357	0.0294	0.0224	0.0149	0.0073
0.45	0.0456	0.0450	0.0433	0.0404	0.0365	0.0317	0.0260	0.0197	0.0131	0.0064
0.50	0.0406	0.0400	0.0385	0.0359	0.0324	0.0280	0.0230	0.0174	0.0115	0.0056
0.55	0.0361	0.0356	0.0342	0.0319	0.0287	0.0248	0.0203	0.0154	0.0102	0.0050
0.60	0.0321	0.0317	0.0304	0.0283	0.0255	0.0220	0.0180	0.0136	0.0090	0.0044
0.65	0.0286	0.0282	0.0270	0.0252	0.0226	0.0195	0.0159	0.0120	0.0079	0.0039
0.70	0.0254	0.0250	0.0240	0.0223	0.0201	0.0173	0.0141	0.0106	0.0070	0.0034
0.75	0.0225	0.0222	0.0213	0.0198	0.0178	0.0153	0.0125	0.0094	0.0062	0.0030
0.80	0.0200	0.0197	0.0189	0.0176	0.0158	0.0136	0.0111	0.0083	0.0055	0.0027
0.85	0.0178	0.0175	0.0168	0.0156	0.0140	0.0120	0.0098	0.0074	0.0049	0.0024
0.90	0.0158	0.0156	0.0149	0.0139	0.0124	0.0107	0.0087	0.0065	0.0043	0.0021
0.95	0.0140	0.0138	0.0132	0.0123	0.0110	0.0095	0.0077	0.0058	0.0038	0.0019
1.00	0.0124	0.0123	0.0117	0.0109	0.0098	0.0084	0.0068	0.0051	0.0034	0.0016
>1.00	0.138α	0.136α	0.130α	0.121α	0.108α	0.093α	0.075α	0.056α	0.037α	0.018α
					$\alpha = e^{-2.4z}$					

TABLE V. $\sum_{+\mu} \frac{J_0(\mu r)}{\mu^4 J_1(\mu)} e^{-\mu z}$

$z \backslash r$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.0550	0.0543	0.0523	0.0490	0.0445	0.0389	0.0323	0.0248	0.0168	0.0083
0.05	0.0490	0.0484	0.0466	0.0436	0.0395	0.0345	0.0285	0.0218	0.0146	0.0072
0.10	0.0437	0.0431	0.0415	0.0388	0.0351	0.0305	0.0252	0.0192	0.0128	0.0063
0.15	0.0389	0.0384	0.0369	0.0345	0.0312	0.0271	0.0223	0.0169	0.0113	0.0055
0.20	0.0347	0.0342	0.0329	0.0307	0.0277	0.0240	0.0197	0.0149	0.0099	0.0049
0.25	0.0308	0.0304	0.0292	0.0272	0.0246	0.0212	0.0174	0.0132	0.0087	0.0043
0.30	0.0274	0.0271	0.0260	0.0242	0.0218	0.0188	0.0154	0.0117	0.0077	0.0038
0.35	0.0244	0.0241	0.0231	0.0215	0.0193	0.0167	0.0136	0.0103	0.0068	0.0033
0.40	0.0217	0.0214	0.0205	0.0191	0.0172	0.0140	0.0121	0.0091	0.0060	0.0029
0.45	0.0193	0.0190	0.0182	0.0169	0.0152	0.0131	0.0107	0.0081	0.0053	0.0026
0.50	0.0171	0.0169	0.0162	0.0150	0.0135	0.0116	0.0095	0.0071	0.0047	0.0023
0.55	0.0152	0.0150	0.0144	0.0133	0.0120	0.0103	0.0084	0.0063	0.0042	0.0020
0.60	0.0135	0.0133	0.0127	0.0118	0.0106	0.0091	0.0074	0.0056	0.0037	0.0018
0.65	0.0120	0.0118	0.0113	0.0105	0.0094	0.0081	0.0066	0.0049	0.0033	0.0016
0.70	0.0106	0.0105	0.0100	0.0093	0.0083	0.0072	0.0058	0.0044	0.0029	0.0014
0.75	0.0094	0.0093	0.0089	0.0083	0.0074	0.0064	0.0052	0.0039	0.0026	0.0012
0.80	0.0084	0.0082	0.0079	0.0073	0.0066	0.0056	0.0046	0.0034	0.0023	0.0011
0.85	0.0074	0.0073	0.0070	0.0065	0.0058	0.0050	0.0041	0.0030	0.0020	0.0010
0.90	0.0066	0.0065	0.0062	0.0058	0.0052	0.0044	0.0036	0.0027	0.0018	0.0009
0.95	0.0058	0.0058	0.0055	0.0051	0.0046	0.0039	0.0032	0.0024	0.0016	0.0008
1.00	0.0052	0.0051	0.0049	0.0045	0.0041	0.0035	0.0028	0.0021	0.0014	0.0007
>1.00	0.057α	0.056α	0.054α	0.050α	0.045α	0.039α	0.031α	0.023α	0.015α	0.007α

$\alpha = e^{-2.4z}$

The above functions have three important characteristics:

I. In each case they consist of the boundary function $V_n(1, z)$ plus other terms; hence the other terms may be regarded as a correction in calculating the field at a general point.

II. The first part of the correction consists of what might, by analogy with electric circuit theory, be called a transient term. The term arises from the discontinuity in the boundary potential and decreases in a modified exponential manner as one leaves the region of the discontinuity.

III. The second part of the correction, applicable only for $z>0$ and $n\neq 0, 1$ consists of odd or even powers of z according as n is odd or even. The correction is necessary because $V=z^n$ does not satisfy Laplace's equation except for $n=0$ and $n=1$.

As an example of the use of the above formulae in calculating potential fields consider an electrode system consisting of two coaxial cylinders, each of unit radius, separated a distance S along the axis (Fig. 1). The coordinate $z=0$ is chosen halfway between the ends of the cylinders; the left-hand cylinder is taken to be at a potential V_a, the right-hand at a potential V_b.

A first approximation to $V(1, z)$ for this case can be taken as

$$V(1,z) \approx \begin{cases} V_a, & z \leq -\frac{s}{2} \\ V_a + \frac{(V_b - V_a)}{s}\left(z + \frac{s}{2}\right), & -\frac{s}{2} \leq z \leq \frac{s}{2}, \\ V_b, & z \geq \frac{s}{2} \end{cases} \tag{19}$$

in which it is assumed that the potential gradient in the gap between the cylinders is constant. Making use of the functions $V_n(1, z)$ as defined in (14), $V(1, z)$ can be expressed in the form

$$V(1,z) \approx V_a + \frac{(V_b - V_a)}{s}\left\{ V_1\left(1, z+\frac{s}{2}\right) - V_1\left(1, z-\frac{s}{2}\right)\right\}. \tag{20}$$

Thus the first approximation to the potential distribution becomes

$$V(r,z) \approx V_a + \frac{(V_b - V_a)}{s}\left\{ V_1\left(r, z+\frac{s}{2}\right) - V_1\left(r, z-\frac{s}{2}\right)\right\}. \tag{21}$$

A better approximation is based on a Legendre polynomial expansion of the potential in the

gap. As is well known[5] the Legendre polynomials $P_n(x)$ provide a series of functions which are orthogonal in the interval $-1<x<1$ and may be used to expand an arbitrary function in this interval. The $P_n(x)$ are polynomials of degree n. The expansion in terms of the polynomials has the valuable property of being the best least-square approximation to the function for a polynomial of degree m, where the series is terminated at $P_m(x)$.

If x in $P_n(x)$ is replaced by $2z/s$ the interval of orthogonality of the polynomials becomes $-s/2<z<s/2$; this interval is the one in which the expansion of $V(1, z)$ is desired. The expansion takes the form

$$V(1,z)=\sum_{n=0}^{\infty} a_n P_n\left(\frac{2}{s}z\right), \quad -\frac{s}{2}<z<\frac{s}{2}, \tag{22}$$

where

$$a_n=\frac{2n+1}{s}\int_{-s/2}^{s/2} V(1,z)P_n\left(\frac{2}{s}z\right)dz. \tag{23}$$

Since, for the particular case considered, $V(1, z)$ is an odd function of z (excepting the constant term) and since $P_n(x)$ is an odd or even function according as n is odd or even, the integral in (23) vanishes for all even n (excepting $n=0$). As an analytical expression for $V(1, z)$ is not known in the interval $-s/2<z<s/2$ the integral in (23) is evaluated numerically, the function $V(1, z)$ having been determined in the interval by measurements made in an electrolytic plotting tank. The results of such measurements and a subsequent analysis for the coefficients a_n are

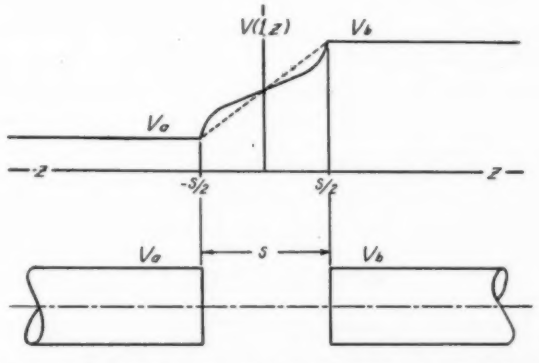

FIG. 1. Boundary potential distribution of cylindrical electrode system.

[5] L. R. Ford, *Differential Equations* (McGraw-Hill Book Company, New York, 1933), pp. 190–197.

TABLE VI.

Separation S	a_0	a_1	a_3
R	0.500	0.38	0.054
$2R$	0.500	0.35	0.049
$3R$	0.500	0.35	0.052
$4R$	0.500	0.36	0.052
$6R$	0.500	0.37	0.057
$8R$	0.500	0.37	0.044

given in Table VI. For convenience the coefficients are based on potentials $V_a=0$ and $V_b=1$.

In terms of the cubic approximation the field at a general point can readily be found. Thus the potential $V(1, z)$ can be approximated by

$$V(1,z)\approx\begin{cases} V_a, & z\leq -\dfrac{s}{2} \\ V_a+(V_b-V_a)\left\{a_0P_0\left(\dfrac{2}{s}z\right)+a_1P_1\left(\dfrac{2}{s}z\right)+a_3P_3\left(\dfrac{2}{s}z\right)\right\}, & -\dfrac{s}{2}<z<\dfrac{s}{2}. \\ V_b, & z\geq\dfrac{s}{2}. \end{cases} \tag{24}$$

Since $P_0(x)=1$, $P_1(x)=x$ and $P_3(x)=\frac{1}{2}(5x^3-3x)$ the potential in the gap can be written

$$V_a+(V_b-V_a)\left\{a_0+\frac{2a_1}{s}z+a_3\left(\frac{20z^3}{s^3}-\frac{3z}{s}\right)\right\}$$

$$=V_a+(V_b-V_a)\left\{a_0+\frac{(2a_1-3a_3)}{s}z+\frac{20a_3z^3}{s^3}\right\}. \tag{25}$$

Replacing z by $[z+(s/2)]-s/2$ and expanding in terms of $[z+(s/2)]$ one obtains

$$V_a+(V_b-V_a)\left\{(a_0-a_1-a_3)+\frac{(2a_1+12a_3)}{s}\left(z+\frac{s}{2}\right)-\frac{30a_3}{s^2}\left(z+\frac{s}{2}\right)^2+\frac{20a_3}{s^3}\left(z+\frac{s}{2}\right)^3\right\}$$

so that in terms of the functions $V_n(1, z)$ the

gap potential can be expressed

$$V_a+(V_b-V_a)\Big\{(a_0-a_1-a_3)\,V_0\Big(1,z+\frac{s}{2}\Big)$$
$$+\frac{(2a_1+12a_3)}{s}V_1\Big(1,z+\frac{s}{2}\Big)-\frac{30a_3}{s^2}V_2\Big(1,z+\frac{s}{2}\Big)$$
$$+\frac{20a_3}{s^3}V_3\Big(1,z+\frac{s}{2}\Big)\Big\}. \quad (26)$$

Similarly, by writing $[z-(s/2)]+s/2$ for z the corresponding expression becomes

$$V_a\cdot V_0\Big(1,z-\frac{s}{2}\Big)+(V_b-V_a)\Big\{(a_0+a_1+a_3)$$
$$\times V_0\Big(1,z-\frac{s}{2}\Big)+\frac{(2a_1+12a_3)}{s}V_1\Big(1,z-\frac{s}{2}\Big)$$
$$+\frac{30a_3V_2}{s^2}\Big(1,z-\frac{s}{2}\Big)+\frac{20a_3V_3}{s^3}\Big(1,z-\frac{s}{2}\Big)\Big\}. \quad (27)$$

The expression (26) represents the potential $V(1,z)$ for $z<s/2$ since for $z<-s/2$ it reduces to V_a. For $z<s/2$ (27) is zero while for $z>s/2$ it is identically equal to (26). Thus subtracting (27) from (26) and adding $V_b\cdot V_0[1,z-(s/2)]$ the desired form of $V(1,z)$ is obtained. The potential at a general point is thus approximated by the formula

$$V(r,z)\approx V_a+(V_b-V_a)\Big\{(a_0-a_1-a_3)$$
$$\times\Big[V_0\Big(r,z+\frac{s}{2}\Big)+V_0\Big(r,z-\frac{s}{2}\Big)\Big]$$
$$+\frac{(2a_1+12a_3)}{s}\Big[V_1\Big(r,z+\frac{s}{2}\Big)-V_1\Big(r,z-\frac{s}{2}\Big)\Big]+$$
$$-\frac{30a_3}{s^2}\Big[V_2\Big(r,z+\frac{s}{2}\Big)+V_2\Big(r,z-\frac{s}{2}\Big)\Big]$$
$$+\frac{20a_3}{s^3}\Big[V_3\Big(r,z+\frac{s}{2}\Big)-V_3\Big(r,z-\frac{s}{2}\Big)\Big]\Big\}. \quad (28)$$

With the aid of Tables II–V the potential field for the electrode system under discussion can now be calculated. The equipotential lines for the case $s=0.2R$ are plotted in Fig. 2. These were obtained from a calculation using the linear approximation, taking $a_0=0.500$, $a_1=0.350$, and $a_3=0.000$. For comparison, data obtained in an electrolytic plotting tank are shown as circled points. The discrepancies can be attributed to inaccuracies of the measurements. The error arising from the linear approximation is greatest in the vicinity of the edges of the electrodes. If greater accuracy is required the higher order approximation may be used.

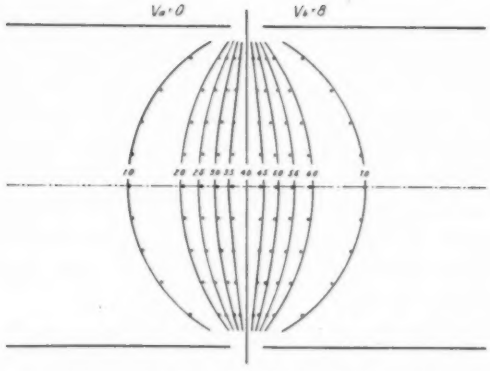

FIG. 2. Equipotential lines of cylindrical electrode system ($V_a=0, V_b=8, s=0.2R$).

The extension of the method to more general cases of axially symmetric electrode systems having any electrode configuration is obvious. The only requirement is that the potential along a cylinder of constant radius—within which Laplace's equation holds—be known with the required accuracy.

The essential principle of the method outlined in this paper for the calculation of fields is that of superposition. Complex boundary potential functions are expressed over short intervals in terms of functions which can be evaluated and the results superposed to obtain the complete solution. Tabulated values have been included to facilitate the use of the method.

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The Operational Theory of Longitudinal Impact

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The differential equations governing the longitudinal displacement and velocity of a uniform rod are identical to those governing the propagation of charge and current along a uniform electrical transmission line. By making use of this analogy between the rod and the transmission line, it is possible to translate some well-known results of electrical circuit theory to mechanical terms. In this manner, and by the use of the operational calculus, interesting and useful results in the theory of longitudinal impact are obtained. The treatment is vastly simpler and more general than the classical theory of Saint-Venant and Boussinesq and some of the results are thought to be new.

I. INTRODUCTION

THE problem considered is that of determining the subsequent behavior of a long, uniform thin rod that is struck at one end by a massive body moving in the direction of the length of the rod. Different states of mobility of the other end of the rod are assumed. The case where the rod is embedded in a viscous medium is also studied.

The classical solution of this problem was obtained by J. Boussinesq[1,2] and Saint-Venant[3] using the general solution of the wave equation that governs the propagation of the displacement of the rod. This method of solution is laborious and difficult to extend to general cases of interest. The method presented in this paper is much simpler and permits more general cases to be studied than is possible by the method of Boussinesq and Saint-Venant.

II. THE EQUATIONS OF MOTION-ELECTRICAL ANALOGY

Consider a uniform rod or bar as shown in Fig. 1.

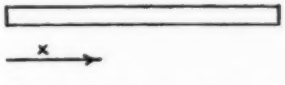

FIG. 1. Uniform rod or bar.

Let us introduce the following notation:

E_y = the Young's modulus of the material of the bar.
ρ = the density of the material of the bar.
t = the time.
x = the longitudinal coordinate measured from one end of the bar.
$u(x, t)$ = the longitudinal displacement of any cross section of the bar during vibration from its position of equilibrium.
$v(x, t)$ = the velocity of any cross section of the bar during vibration.
$F(x, t)$ = the longitudinal compressive force acting across any section for the bar.
A = the cross-sectional area of the bar.

It is easy to show[4] that the displacement, $u(x, t)$ satisfies the wave equation,

$$\partial^2 u/\partial t^2 = c_0^2 \partial^2 u/\partial x^2 \tag{1}$$

where

$$c_0^2 = E_y/\rho \tag{2}$$

and x is measured as in Fig. 1. The compressive force $F(x, t)$ satisfies the equation,

$$F = -AE_y \partial u/\partial x. \tag{3}$$

By differentiating (1) with respect to x and substituting (3) into the resulting equation, we obtain:

$$\partial^2 F/\partial t^2 = c_0^2 \partial^2 F/\partial x^2. \tag{4}$$

That is, the force $F(x, t)$ satisfies the same wave equation that is satisfied by the displacement, $u(x, t)$. The velocity, $v(x, t)$ of any point

[1] J. Boussinesq, *Applications des Potentiels à l'étude de l'équilibre et du Mouvement des Solides Elastiques* (Paris 1885), p. 508.
[2] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge, 1934), p. 431.
[3] Barre de Saint-Venant, *Theorie de l'elasticité des corps solides* (Paris, 1883), p. 307.
[4] S. Timoshenko, *Vibration Problems in Engineering* (D. Van Nostrand, 1937).

of the rod is defined by,

$$v(x, t) = \partial u/\partial t. \tag{5}$$

Differentiating (1) with respect to t and substituting from (5), we obtain,

$$\partial^2 v/\partial t^2 = c_0^2 \partial^2 v/\partial x^2. \tag{6}$$

Differentiating, (3) with respect to t, we have

$$\frac{\partial F}{\partial t} = -(AE_y)\frac{\partial}{\partial t}(\partial u/\partial x) = -(AE_y)\frac{\partial}{\partial x}(\partial u/\partial t)$$

$$= -(AE_y)(\partial v/\partial x). \tag{7}$$

Equation (1) may be written in the form,

$$\rho \partial^2 u/\partial t^2 = E_y \partial^2 u/\partial x^2. \tag{8}$$

Substituting (3) and (5) into (8), we obtain,

$$\rho \frac{\partial v}{\partial t} = -\frac{1}{A}\frac{\partial F}{\partial x}. \tag{9}$$

Equations (7) and (9) may be written in the form,

$$\frac{1}{AE_y}\frac{\partial F}{\partial t} = -\frac{\partial v}{\partial x}, \tag{10}$$

$$\rho A \partial v/\partial t = -\partial F/\partial x. \tag{11}$$

Let us now consider a dissipationless electrical transmission line as shown in Fig. 2.

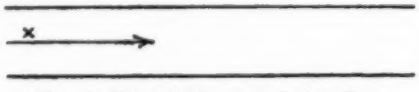

FIG. 2. Electrical transmission line.

Let L = the inductance per unit length of the transmission line.
C = the capacitance per unit length of the transmission line.
$e(x, t)$ = the potential difference between the line conductors.
$i(x, t)$ = the line current.
x = the space coordinate measured from one end of the line.
t = the time.

It may be shown[5] that the equations governing the propagation of potential and current along the line are,

$$C\partial e/\partial t = -\partial i/\partial x, \tag{12}$$

$$L\partial i/\partial t = -\partial e/\partial x. \tag{13}$$

The Eqs. (10) and (11) are of exactly the same form as (12) and (13). We thus see that we may write the following relations,

$$F(x, t) \doteqdot e(x, t), \tag{14}$$

$$v(x, t) \doteqdot i(x, t), \tag{15}$$

$$1/AE_y \doteqdot C, \tag{16}$$

$$(\rho A) = m \doteqdot L. \tag{17}$$

The symbol $\doteqdot$ expresses the fact that the two quantities are analogous and m is the mass per unit length of the rod. By the above analogy the mechanical problem may be translated into an electrical problem. Many well-known results in the theory of electrical networks may be used. To do this, it is only necessary to replace masses by inductances, springs by suitable condensers, and frictional forces of the coulomb type by electrical resistances.

The problem of determining the behavior of a rod of length s and fixed at one end when struck at the other end by a mass m_0 moving with a velocity v_0 as shown in Fig. 3, is analogous to the

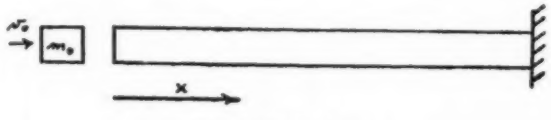

FIG. 3. Fixed bar.

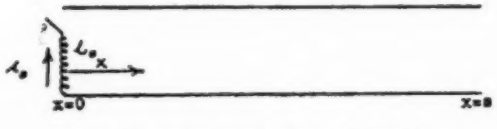

FIG. 4. Electrical equivalent of fixed bar.

problem of determining the current and potential distribution of an open-ended transmission line when an inductance L_0 carrying an initial current i_0 is suddenly connected to the other end of the line. In order to facilitate the translation of electrical to mechanical quantities, the following table of analogs, is useful.

[5] E. A. Guillemin, *Communication Networks* (John Wiley, 1935), Vol. 2, Chap. 1.

TABLE I. Table of analogs.

Rod	Transmission line
displacement, $u(x, t)$	charge distribution, $q(x, t)$
velocity, $v(x, t)$	current, $i(x, t)$
compressive force, $f(x, t)$	potential distribution, $e(x, t)$
lumped mass, m_0	lumped inductance, L_0
distributed mass/length $m=(\rho A)$	inductance/length, L
compliance/length, $1/AE_y$	capacitance/length, C
velocity of wave propagation, $c_0=(E_y/\rho)^{\frac{1}{2}}$	velocity of wave propagation, $v_0=1/(LC)^{\frac{1}{2}}$
initial velocity, v_0	initial current, i_0
length or rod, s	length of transmission line, s
ratio of striking mass to mass of rod, $n=m_0/ms$	ratio of inductance L_0 to line inductance $n=L_0/sL$
damping coefficient per unit length, R_m	resistance/length, R
dashpot damping coefficient, R_d	resistance, R_0
spring stiffness, k_0	elastance, S_0
mechanical surge impedance of rod, c_0m	surge impedance, $Z_0=(L/C)^{\frac{1}{2}}$

III. THE OPERATIONAL EQUATIONS OF THE TRANSMISSION LINE

By the above analogs, the study of longitudinal impact of rods or bars is reduced to the study of transients along a transmission line which are initiated by certain initial conditions. The simplest mathematical technique to use in solving problems of this character is the operational calculus.

As is well known, the modern theory of the operational calculus is based upon the Fourier-Mellin integral equations.[6] This set of equations states that if,

$$g(p)=p\int_0^\infty e^{-pt}h(t)dt, \tag{18}$$

then,

$$h(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} g(p)e^{pt}dp/p, \text{ where } j=\sqrt{-1} \tag{19}$$

and c is a real constant of such magnitude that the poles of the integrand lie entirely to the left of the path of integration in the complex p plane. For brevity, let us use the notation,

$$g(p)=Lh(t), \tag{20}$$

$$h(t)=L^{-1}g(p), \tag{21}$$

to express, symbolically, the relations (18) and (19).

Let us now consider the transmission line of Fig. 5.

[6] L. A. Pipes, "The operational calculus," J. App. Phys. 10, 172, 258, 301 (1939).

Z_s and Z_r represent the impedances of linear networks connected to the sending and receiving ends of the line. At the sending end, an electromotive force $e_g(t)$ which may be a general function of time is impressed. If the line is dissipationless and has an inductance L and capacitance C per unit length, then the potential $e(x, t)$ and the current $i(x, t)$ satisfy the Eqs. (12) and (13) along the line. The equations

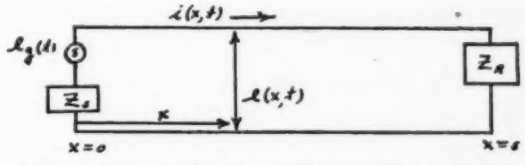

FIG. 5. Transmission line with general impedances.

$$e_g=Z_si(0, t)+e(0, t) \tag{22}$$

and

$$e(s, t)=Z_ri(s, t) \tag{23}$$

must be satisfied at the sending and receiving ends, respectively. If we now introduce the transforms,

$$E_g(p)=Le_g(t), \tag{24}$$

$$E(x, p)=Le(x, t), \tag{25}$$

$$I(x, p)=Li(x, t), \tag{26}$$

and make use of Eqs. (12), (13), (22), and (23) we obtain,[6]

$$E(x, p)=\frac{Z_0E_g[e^{a(s-x)}+r_re^{-a(s-x)}]}{(Z_s+Z_0)(e^{as}-r_sr_re^{-as})}, \tag{27}$$

$$I(x, p)=\frac{E_g[e^{a(S-x)}-r_re^{-a(s-x)}]}{(Z_s+Z_0)(e^{as}-r_sr_re^{-as})}, \tag{28}$$

where

$$a=p(LC)^{\frac{1}{2}}, \tag{29}$$

$$Z_0=(L/C)^{\frac{1}{2}}, \tag{30}$$

$$r_s=(Z_s-Z_0)/(Z_s+Z_0), \tag{31}$$

$$r_r=(Z_r-Z_0)/(Z_r+Z_0). \tag{32}$$

These are the well-known equations for the transforms of the potential $E(x, p)$ and current $I(x, p)$, respectively, of a dissipationless line. The quantities r_s and r_r are the operational reflection coefficients and Z_0 is the characteristic or surge impedance of the line.

If the character of the impressed electromotive force $e_g(t)$ is known, then it is possible to com-

pute its transform $E_g(p)$ and the potential and current distribution may be determined by finding the inverse transforms of $E(x, p)$ and $I(x, p)$ as given by (27) and (28). In the theory of impact, we are in general interested in the so-called wave solution of the problem. That is, the impact of the mass m_0 on the rod produces a wave of force and deflection that travels along the rod with a certain velocity and is reflected at the end of the rod in a certain manner. To obtain the character of these waves, let us divide the numerator and denominator of (27) and (28) by e^{as}. We then obtain,

$$E(x, p)=\frac{Z_0E_g[e^{-as}+r_re^{-a(2s-x)}]}{(Z_0+Z_s)(1-r_sr_re^{-2as})}, \quad (33)$$

$$I(x, p)=\frac{E_g[e^{-as}-r_re^{-a(2s-x)}]}{(Z_0+Z_s)(1-r_sr_re^{-2as})}. \quad (34)$$

The second factor in the denominator of (33) and (34) may be expanded in the form,

$$[1-r_sr_re^{-2as}]^{-1}=\sum_{n=0}^{\infty}(r_sr_r)^ne^{-2asn}. \quad (35)$$

Substituting this into (33) and (34), we obtain,

$$E(x, p)=\frac{Z_0E_g}{(Z_s+Z_0)}\sum_{n=0}^{\infty}(r_sr_r)^ne^{-a(x+2sn)}$$

$$+\frac{Z_0E_gr_r}{(Z_s+Z_0)}\sum_{n=0}^{\infty}(r_sr_r)^ne^{-a(2s+2sn-x)}, \quad (36)$$

$$I(x, p)=\frac{E_g}{(Z_s+Z_0)}\sum_{n=0}^{\infty}(r_sr_r)^ne^{-a(x+2sn)}$$

$$-\frac{E_gr_r}{(Z_s+Z_0)}\sum_{n=0}^{\infty}(r_sr_r)^ne^{-a(2s+2sn-x)}. \quad (37)$$

The expressions (36) and (37) have simple physical interpretations. First let us consider the case where the terminal impedance of the system Z_r satisfies the equation

$$Z_r=Z_0=(L/C)^{\frac{1}{2}}. \quad (38)$$

In this case, we have from (32),

$$r_r=0. \quad (39)$$

In such a case (36) and (37) reduce to the simple forms,

$$E(x, p)=Z_0E_ge^{-ax}/(Z_s+Z_0), \quad (40)$$

$$I(x, p)=E_ge^{-ax}/(Z_s+Z_0). \quad (41)$$

These expressions are the transforms of waves of potential and current progressing along the line with a velocity of,

$$V_0=1/(LC)^{\frac{1}{2}}. \quad (42)$$

These waves are not reflected from the receiving end as a consequence of the relation $r_r=0$. If $r_r\neq0$, the series of terms in (36) and (37) are the transforms of waves produced by successive reflections from the ends of the line.

IV. ROD FIXED AT ONE END AND STRUCK LONGITUDINALLY AT THE OTHER

Let us consider the case of Fig. 3. In this case, the rod is struck by a mass m_0 moving with velocity v_0 and the other end of the rod is fixed. To solve the problem of determining the subsequent state of affairs, we consider the electrical analog of the rod as shown in Fig. 4.

In order to use the general results (36) and (37) for the case of the inductance L_0 carrying a current i_0 and suddenly switched on to the open-circuited line, we place,

$$E_g=L_0pi_0, \quad (43)$$

$$Z_s=L_0p, \quad (44)$$

$$Z_r=\infty. \quad (45)$$

Hence we have,

$$r_r=(Z_r-Z_0)/(Z_r+Z_0)=1, \quad (46)$$

$$r_s=(L_0p-Z_0)/(L_0p+Z_0), \quad (47)$$

$$E_g/(Z_s+Z_0)=L_0pI_0/(L_0p+Z_0)$$

$$=pI_0/(p+Z_0/L_0). \quad (48)$$

Then by (36), we have:

$$E(x, p)=\frac{Z_0pi_0}{p+Z_0/L_0}[e^{-ax}+r_re^{-a(2s-x)}$$

$$+r_sr_re^{-a(2s+x)}+r_rr_rr_se^{-a(4s-x)}+\cdots] \quad (49)$$

for the transform of the potential.

To interpret this result, let us consider the

first term of (49).

$$\frac{Z_0 p i_0 e^{-ax}}{(p+Z_0/L_0)}=\frac{Z_0 p i_0 \exp(-xp/V_0)}{(p+b)}, \quad (50)$$

where,

$$b=Z_0/L_0, \quad (51)$$

$$V_0=1/(LC)^{\frac{1}{2}}. \quad (52)$$

From the theory of Laplacian transforms,[6] we have the following inverse transform,

$$L^{-1}\frac{Z_0 p i_0 \exp(-px/V_0)}{(p+b)}$$

$$=\begin{cases}0 & t<x/V_0\\ i_0 Z_0 \exp(-b(t-x/V_0)) & t>x/V_0.\end{cases} \quad (53)$$

The first term of (49) thus represents the inverse transform of a traveling wave of electromotive force which progresses along the line with a velocity of V_0. It is convenient to introduce the quantity,

$$L_0/Ls=n. \quad (54)$$

n thus represents the ratio of the lumped inductance L_0 to the entire inductance of the line. With this notation, we may write,

$$i_0 Z_0 \exp[-b(t-x/V_0)]$$

$$=i_0 Z_0 \exp(x-tV_0)/ns]. \quad (55)$$

To interpret the mechanical significance of the solution, it is necessary only to examine Table I. From this, we have,

$$i_0 Z_0 \exp[-(t-x/V_0)b]$$

$$\doteqdot\begin{cases}0 & t<x/c_0\\ v_0 c_0 m \exp[(x-c_0 t)/ns] & t>x/c.\end{cases} \quad (56)$$

That is, the effect of the impact is to create a pressure wave which progresses along the rod to the fixed end with a velocity c_0. Let us now examine the inverse transform of the second term of (49). Since r_r is equal to 1, this term is,

$$L^{-1}\frac{Z_0 p i_0 e^{-a(2s-x)}}{(p+b)}$$

$$=\begin{cases}0 & t<(2s-x)/V_0\\ i_0 Z_0 \exp[-b[t-(2s-x)/V_0]] & t>(2s-x)/V_0.\end{cases} \quad (57)$$

By Table I, we have:

$$L^{-1} i_0 Z_0 \exp[-b[t-(2s-x)/V_0]]$$

$$\doteqdot v_0 c_0 m \exp[[(2s-x)-c_0 t]/ns]. \quad (58)$$

We thus see that the second term of (49) is the transform of a wave of compression traveling back from the fixed end of the rod with a velocity c_0. In the same manner, it may be demonstrated that the inverse transform of the third term of (49) is given by,

$$\frac{Z_0 p i_0 r_s e^{-a(2s+x)}}{(p+b)}$$

$$=\begin{cases}0 & t<(2s+x)/V_0\\ i_0 Z_0 \exp[-b[t-(2s+x)/V_0]] & \\ \quad -2i_0 Z_0 b[t-(2s+x)/V_0] & \\ \quad \times\exp[-b[t-(2s+x)/V_0]] & \\ & t>(2s+x)/V_0.\end{cases} \quad (59)$$

By Table I, we have,

$$i_0 Z_0 \exp[-b[t-(2s+x)/V_0]$$

$$-2i_0 Z_0 b[t-(2s+x)/V_0]$$

$$\times\exp[-b[t-(2s+x)/V_0]]$$

$$\doteqdot v_0 c_0 m \exp[[(2s+x)-c_0 t]/ns]$$

$$\times\{1+2[(2s+x)-c_0 t]/ns\}. \quad (60)$$

This represents a wave of compression traveling from the end of the rod that was struck toward the fixed end with a velocity c_0. If the striking body becomes attached to the rod, we may calculate the subsequent waves of pressure by determining the inverse transforms of the various terms of (49). However, if the bodies remain detached, the solution continues to hold so long as there is positive pressure between the rod and the striking body.

To examine when the pressure ceases to be positive, we place $x=0$ in the wave solutions. If we let f_0 be the compressive force at $x=0$, we have,

$$f_0=v_0 c_0 m \exp(-c_0 t/ns)$$

$$\text{for } 0<t<2s/c_0. \quad (61)$$

This expression is positive. However, in the

range, $2s/c_0<t<4s/c_0$, the compressive force is given by,

$$f_0=v_0c_0m\exp\left[-c_0t/(ns)\right]\times\left[1+2e^{2/n}\left(1-\frac{c_0t-2s}{ns}\right)\right]. \quad (62)$$

This expression vanishes, when,

$$2c_0t/ns=4/n+2+e^{-2/n}. \quad (63)$$

It is possible for this equation to have a root in the interval $2s/c_0<t<4s/c_0$ if,

$$2+e^{-2/n}<4/n. \quad (64)$$

It may be shown that the equation,

$$2+e^{-2/n}=4/n \quad (65)$$

has a root lying between $n=1$ and $n=1.73$. Hence if $n<1.73$ the impact ceases in the above interval and the time at which it ceases is given by,

$$t=s(2+n+ne^{-2/n}/2)/c_0. \quad (66)$$

To interpret the wave solution for the velocity, it is necessary only to compute the inverse transforms of $I(x, p)$ when the values given by (43), (44) and (45) are substituted into (37).

V. ROD FREE AT ONE END AND STRUCK LONGITUDINALLY AT THE OTHER

Let us now consider the case of Fig. 6. In this

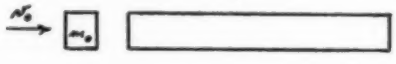

FIG. 6. Free bar.

case, the bar is free to move and the electrical analog of the system is an inductance connected to a short-circuited line at $t=0$, as shown in Fig. 7.

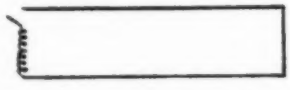

FIG. 7. Short-circuited line.

In order to treat this case, it is necessary only to realize that we now have,

$$Z_r=0. \quad (67)$$

and hence,

$$r_r=-1. \quad (68)$$

The relations (43), and (44) still hold. From (49), we have, on placing $x=0$ and $r_r=-1$.

$$E(0, p)=\frac{Z_0pi_0}{(p+b)}\left[1-e^{-2as}-r_se^{-2as}+e^{-4as}-\cdots\right]. \quad (69)$$

This is the transform of the potential at $x=0$. In view of the transforms (53), (57), and (59) it is easily seen that the potential, and hence the mechanical compressive force becomes negative at $x=0$ at the instant,

$$t=2s/c_0. \quad (70)$$

This means that the impact produces a wave of compression which is reflected from the free end as a wave of extension that causes the impact to cease. It is interesting to compute the velocity of the rod after impact.

To do this, we substitute (43), (44), (47), and (68) into the transform of the current given by (37) and obtain,

$$I(x, p)=\frac{pi_0}{(p+b)}\left[e^{-ax}+e^{-a(2s-x)}-r_se^{-a(2s+x)}+\cdots\right]. \quad (71)$$

From (53), we have

$$L^{-1}\frac{pi_0e^{-ax}}{(p+b)}=\begin{cases}0 & t<x/V_0\\ i_0e^{(x-V_0t)/ns} & t>x/V_0.\end{cases} \quad (72)$$

Hence the mechanical analog of the first term of (71) is the transform of a wave of velocity given by (72). From (57), we have,

$$\frac{pi_0e^{-a(2s-x)}}{(p+b)}=\begin{cases}0 & t<(2s-x)/V_0\\ i_0\exp\left[\left[(2s-x)-V_0t\right]/ns\right] & t>(2s-x)/V_0.\end{cases} \quad (73)$$

The mechanical analog of the right member of (73) represents a wave of velocity reflected from the free end and traveling toward the struck end with a velocity c_0. At $t_0=2s/c_0$ the velocity of the rod is given by,

$$v(x, t_0)=v_0\left[e^{-2/n}e^{x/ns}+e^{-x/ns}\right] \quad (74)$$

or if we let

$$x=s-y, \quad (75)$$

this reduces to

$$2v_0e^{-1/n}\cosh(y/ns). \tag{76}$$

It is thus obvious that the rod rebounds vibrating.

VI. ROD TERMINATED BY A DASH POT

A very interesting case arises when the rod is struck at one end and the other end is connected to a dash pot of a certain strength, as shown in Fig. 8.

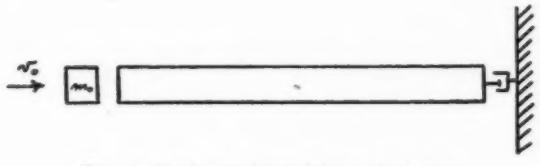

FIG. 8. Rod terminated by a dash pot.

The electrical analog of this is the transmission line terminated by a resistance as shown in Fig. 9.

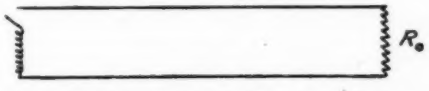

FIG. 9. Line terminated by a resistance.

In this case, we have,

$$Z_r=R_0 \tag{77}$$

and

$$r_r=(R_0-Z_0)/(R_0+Z_0). \tag{78}$$

It is thus seen that the reflection coefficient r_r is a numerical factor less than one. Let us consider the case where the electrical transmission line is terminated in a resistance whose value is equal to the characteristic impedance of the line. That is, the case where

$$R_0=Z_0. \tag{79}$$

In the mechanical analog, this means that one end of the rod encounters a resistance proportional to its velocity of magnitude,

$$R_m=c_0m. \tag{80}$$

In this case, we have,

$$r_r=0, \tag{81}$$

and the solution is vastly simplified because we do not have any waves reflected from the dashpot end. In this case, the transforms of the potential and current are given by,

$$E(x,p)=Z_0pi_0e^{-ax}/(p+b), \tag{82}$$

$$I(x,p)=pi_0e^{-ax}/(p+b). \tag{83}$$

The transform of the charge distribution is given by,

$$Q(x,p)=I(x,p)/p=i_0e^{-ax}/(p+b). \tag{84}$$

Interpreting the inverse transforms of these quantities in terms of their mechanical analogues, we obtain the following expressions for the compressive force, velocity, and displacement of the rod,

$$f(x,t)=\begin{cases}0 & t<x/c_0\\ v_0c_0m\exp[(x-c_0t)/ns] & t>x/c_0,\end{cases} \tag{85}$$

$$v(x,t)=\begin{cases}0 & t<x/c_0\\ v_0\exp[(x-c_0t)/ns] & t>x/c_0,\end{cases} \tag{86}$$

$$u(x,t)=\begin{cases}0 & t<x/c_0\\ [1-\exp(x-c_0t)/ns]v_0sn/c_0 & t>x/c_0.\end{cases} \tag{87}$$

From (87) we have the result that the displacement of the struck end is given by

$$u(0,t)=[1-\exp(-c_0t/ns)]v_0sn/c_0. \tag{88}$$

It is thus evident that as a consequence of the impact, the struck end moves a distance snv_0/c_0.

VII. INFINITE ROD UNDER ACTION OF A SPRING

To illustrate how the general equations may be used to analyze special cases, let us consider the case of Fig. 10. In this case, a spring of

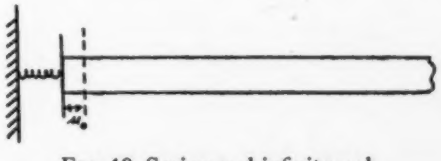

FIG. 10. Spring and infinite rod.

spring constant k_0 is deflected from its position of equilibrium by a distance u_0 so that it has an initial potential energy of $k_0u_0^2/2$. The spring

is released at $t=0$ and its potential energy is transferred to an infinite rod.

The electrical analog of this system is a condenser discharging into an infinite transmission line as shown in Fig. 11.

FIG. 11. Condenser discharging into infinite line.

In the electrical analog, the electric energy of the condenser is $S_0Q_0^2/2$, where S_0 is the elastance of the condenser and Q_0 is its initial charge. In order to determine the waves of potential and current we place,

$$r_r=0, \tag{88}$$

$$E_g=S_0Q_0, \tag{89}$$

$$Z_s=S_0/p, \tag{90}$$

in the general Eqs. (36) and (37). We thus obtain,

$$E(x,p)=S_0Q_0Z_0e^{-ax}/(Z_0+S_0/p), \tag{91}$$

$$I(x,p)=S_0Q_0e^{-ax}/(Z_0+S_0/p), \tag{92}$$

$$Q(x,p)=I(x,p)/p. \tag{93}$$

Computing the inverse transforms of these expressions, we obtain,

$$e(x,t)=\begin{cases}0 & t<x/V_0\\ S_0Q_0e^{S_0C(x-tV_0)} & t>x/V_0,\end{cases} \tag{94}$$

$$i(x,t)=\begin{cases}0 & t<x/V_0\\ \dfrac{S_0Q_0}{Z_0}\exp\left[S_0C(x-tV_0)\right] & t>x/V_0,\end{cases} \tag{95}$$

$$q(x,t)=\begin{cases}0 & t<x/V_0\\ Q_0[1-\exp\left[S_0C(x-tV_0)\right] & t>x/V_0.\end{cases} \tag{96}$$

These expressions may be easily translated into mechanical language by the table of analogs.

VIII. LONG ROD EMBEDDED IN A VISCOUS MEDIUM

A problem of some interest, is that of determining the character of the waves propagated along a rod that is embedded in a medium that impedes its motion with a force proportional to its velocity. To simplify the problem, let us consider the case of an infinite rod or one that is terminated by a dash pot of such a strength that no waves are reflected. We shall consider the case in which a force f_0 is suddenly applied to the rod as shown in Fig. 12.

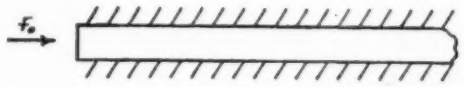

FIG. 12. Rod embedded in a viscous medium.

The electrical equivalent of this system is given in Fig. 13.

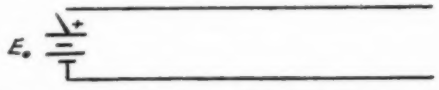

FIG. 13. Electrical transmission line with resistance.

In this case, the electrical transmission line has a resistance R per unit length, and hence the operational propagation constant, a, is given by

$$a=[(Lp+R)(Cp)]^{\frac{1}{2}} \tag{97}$$

instead of by (29).

To determine the waves of potential and current, we place,

$$r_r=0, \tag{98}$$

$$E_g=E_0, \tag{99}$$

$$Z_s=0, \tag{100}$$

in the general Eqs. (36), and (37). We then have,

$$E(x,p)=E_0e^{-ax}, \tag{101}$$

$$I(x,p)=E_0e^{-ax}/Z_0. \tag{102}$$

In this case the computation of the inverse transforms of (101) and (102) is a matter of some complexity, because of the complexity of the operational propagation constant a and the operational characteristic impedance Z_0 which is now given by,

$$Z_0=[(Lp+R)/Cp]^{\frac{1}{2}}. \tag{103}$$

The required inverse transforms may be computed by the use of the basic contour integral of Eq. (19). To facilitate the writing,

let us place,

$$\alpha = R/2L. \tag{104}$$

We then have,

$$a = (LC)^{\frac{1}{2}}[p(p+R/L)]^{\frac{1}{2}} = [p(p+2\alpha)/V_0]^{\frac{1}{2}}. \tag{105}$$

The inverse transform of (101) is then given by,[7]

$$E_0 e^{-ax} = \begin{cases} 0 & t < x/V_0 \\ E_0\left[e^{-\alpha x/V_0} + \dfrac{\alpha x}{V_0}\displaystyle\int_{x/V_0}^{t} \frac{e^{-\alpha u} I_1[\alpha(u^2 - x^2/V_0^2)^{\frac{1}{2}}]du}{(u^2 - x^2/V_0^2)^{\frac{1}{2}}}\right] & \\ & t > x/V_0, \end{cases} \tag{106}$$

where $I_1(y)$ is the modified Bessel function of the first order and argument y. The transform (106) shows that a wave of potential or a wave of compressive force in the mechanical case arrives at a point x from the transmitting end at a time $t_0 = x/V_0$. At that instant, the potential or compressive force rises suddenly to the value $E_0 \exp(-x/V_0)$. The compressive force wave thus travels along the rod with a vertical front whose height steadily decreases with an increase in x. As time progresses, the potential assumes the value E_0 throughout.

To compute the inverse transform of the current, we write,

$$I(x, p) = E_0 e^{-ax}/Z_0 = \frac{E_0(C/L)^{\frac{1}{2}} p \exp[-x[(p+2\alpha)p]^{\frac{1}{2}}/V_0]}{[(p+2\alpha)p]^{\frac{1}{2}}}. \tag{107}$$

[7] N. W. McLachlan, *Complex Variable and Operational Calculus* (Cambridge, 1939), pp. 216–218.

The inverse transform of this expression is given by,[7]

$$L^{-1}I(x, p) = \begin{cases} 0 & t < x/V_0 \\ E_0(C/L)^{\frac{1}{2}} e^{-\alpha t} I_0 & \\ \quad \times[\alpha(t^2 - x^2/V_0^2)^{\frac{1}{2}}] & t > x/V_0. \end{cases} \tag{108}$$

From (108) it is seen that in the analogous mechanical case, a wave of velocity arrives at a distance x from the transmitting end at a time $t_0 = x/V_0$. At this time the current rises suddenly to the value $E_0[(C/L)]^{\frac{1}{2}} \exp(-\alpha x/V_0)$. As time progresses, the current finally assumes the value 0. To obtain the mechanical displacement of the rod, it is necessary to compute the analogous quantity,

$$q(x, t) = \int_0^t i(x, t)dt. \tag{109}$$

This expression gives the wave of displacement that progresses along the rod when interpreted in terms of the various mechanical analogs.

IX. CONCLUSION

It is apparent from the above examples that treating the problems of longitudinal impact of rods in terms of their electrical analogs vastly simplifies the analysis. More complicated terminal conditions than those treated in this discussion may be analyzed by introducing proper terminal impedances in the basic Eqs. (36) and (37). The classical theory of Saint-Venant and Boussinesq is difficult to extend to general cases. The analysis presented in this paper emphasizes the utility and simplicity of operational methods in problems dealing with impulsive forces.

Energy and Energy Flow in the Electromagnetic Field

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(Received April 17, 1942)

A new method for deriving the usual Poynting vector is given. This method also yields other equally valid energy flow vectors with the same or other equally valid postulated electromagnetic energy densities. Examples of such alternative Poynting vectors with their associated energy densities are given. A definition is given which distinguishes between the conduction and displacement currents in matter. It is shown that by addition of a term, the postulated energy flow widely used by electrical power engineers and based on wattmeter readings becomes a valid alternative Poynting vector.

1. Poynting's theory of the energy streaming in an electromagnetic field is usually based on the interpretation of the various terms in the following equation, derived directly from Maxwell's equations:

$$\begin{aligned}-\operatorname{div}\frac{c}{4\pi}[E\times H]&=-\frac{c}{4\pi}H\cdot\operatorname{curl}E+\frac{c}{4\pi}E\cdot\operatorname{curl}H\\&=\frac{c}{4\pi}H\cdot\left(\frac{1}{c}\frac{\partial B}{\partial t}\right)+E\cdot I_t\\&=\frac{1}{4\pi}H\cdot\frac{\partial B}{\partial t}+E\cdot I_t.\end{aligned}\tag{1}$$

In this equation, I_t is the total current density, i.e., the sum of the conduction, convection, and displacement current densities, and perhaps best defined as $(c/4\pi)$ curl H. The other vectors have the significance usually attached to them in Maxwell's theory.

By examining the meaning of the terms on the right-hand side of this equation in various special cases, it may be made plausible that these terms represent the rate of increase of an electromagnetic energy density to be assigned to the point in the empty or material-occupied space being considered, plus the rates of increase of other recognizably localized energy densities such as heat, chemical energy, etc., which are to be regarded as due to the electromagnetic field, plus the rate of doing work of certain ponderomotive forces acting on moving matter per unit volume at the point in question. These ponderomotive forces are regarded as arising from, or being due to, the action of the electromagnetic field on charged, electrically polarized, or magnetized matter. With this interpretation of the right-hand side of this equation, an appropriate meaning may be given to the Poynting vector, $(c/4\pi)[E\times H]$, which appears on the left-hand side of the equation by postulating that it represents a flow of "electromagnetic" energy in space or matter.

It is well known that this Poynting vector is not uniquely defined by Eq. (1), but any other vector with an equal divergence can equally well, according to Eq. (1), claim to represent a flow of electromagnetic energy. The method of deriving energy flow vectors given in this paper exhibits these alternative Poynting vectors with a kind of physical interpretation.

It is probably fairly evident, also, that the success of the Poynting vector in Eq. (1) depends on the choice of the electromagnetic energy density distributed through space. It is also clear that this choice of energy density is to a considerable extent arbitrary, since only its total quantity, integrated over all space may be said to be observable, so that any other assumed density, so long as it gives the same total when integrated over all space, may be regarded as equally valid. Of course, such other assumed electromagnetic energy density will have corresponding to it a valid energy-flow vector which will differ from the Poynting vector even in its divergence. This paper shows how such other energy flow vectors may be constructed.

2. The kinds of special cases which are considered as establishing the interpretation of the terms in (1) in general can be reduced to cases where the changes in the electromagnetic field are produced by the motion of electric charges

alone. By this we mean that energy of a mechanical form changes in magnitude at a rate given by the summation of $\rho E \cdot v$ over all the charged matter, where ρ is the charge density, E the electric field intensity, and v the velocity of the charged matter, and that no changes in any other recognizable form of energy can be discerned, other than the postulated electromagnetic energy which is being calculated and the appearance of heat which must be removed if the temperature at any point is to remain unchanged. The rate of change of the postulated electromagnetic energy is set equal to the rate of increase in mechanical energy due to the motion of the charges plus the rate of development of heat. The formulae for the electromagnetic energy change calculated for these special cases are then assumed to be universally valid.

Further independent assumptions are then made as to the distribution of this energy in space. The formulae derived in the special cases are, or may be converted into, integrals over space of functions of the field vectors, i.e., E, D, H, and B. These functions of the field vectors are then postulated to be the electromagnetic energy density in the special cases considered, and also, so far as they may be given meaning, in general cases. Where the formulae derived for the special cases are not applicable, the Poynting vector is used to define an electromagnetic energy density by postulating that an energy density function T exists defined by

$$\frac{\delta T}{\delta t}+\frac{\delta Q}{\delta t}=-\operatorname{div}\frac{c}{4\pi}[E\times H], \tag{2}$$

where $\delta/\delta t$ is the convective derivative for moving matter, and $\delta Q/\delta t$ is the rate of appearance of heat per unit volume of matter. Thus, for example, Eq. (2) for magnetic matter, in general, makes the energy density, T_m, be given by

$$\frac{\delta T_m}{\delta t}+\frac{\delta Q}{\delta t}=\frac{1}{4\pi}H\cdot\frac{\delta B}{\delta t}. \tag{3}$$

Similarly for dielectric matter, the electric energy density is defined by

$$\frac{\delta T_e}{\delta t}+\frac{\delta Q}{\delta t}=E\cdot I_t=\frac{1}{4\pi}E\cdot\frac{\delta D}{\delta t}+E\cdot I_c, \tag{4}$$

where I_c is the conduction current density.

3. In this paper we consider not special cases, but proceed at once to the most general electromagnetic field. However, for simplicity, we shall consider only fields with matter at rest, although the method is readily extended to matter in motion. We take an arbitrary volume of this electromagnetic field, enclose it in a geometric closed surface, and define an energy flow through this surface in the following way.

We suddenly create a narrow empty shell following the surface. Within this shell we place means for observing the defined energy flow. Thus the definition we are developing for energy flow uses a procedure similar to that frequently used for defining D and B within matter.

The requirement for "suddenness" in the creation of the empty shell is for the purpose of distinguishing between the displacement and conduction current. In such a "suddenly" created shell the normal component of electric force will equal the normal component of D. The rate of change of the normal component of electric force will be 4π times I_t, the total current density.

In the empty shell we place an inner shell of magnetic material of infinite permeability. This magnetic shell we expect to cut off all interplay or interchange of energy between the electromagnetic field inside the shell and the field outside.

To compensate for the disturbing effect of the magnetic shell on the inner field, we place charges between the shell and the field in proper amounts, and give them such motions that the field inside is the same and for some time continues to be the same as if the shell space had not been created. If we like, we may also place moving charges outside the magnetic shell to keep the external field unchanged.

The rate at which work is done by the moving charges is then defined as the energy flow out through the surface from the inner field, with due consideration of possible changes in surface energy density at the shell itself.

The creation of the empty shell, because of its infinitesimal thickness, may perhaps be regarded as involving a negligible amount of energy in its volume. However, where the shell passes through matter, discontinuities are produced in the field vectors, and the possibility must be considered

that a surface distribution of energy must be assigned to these discontinuities.

Likewise, the bringing in of the infinitely permeable material to make the inner magnetic shell, and the bringing in of the compensating charges, presumably from infinitely remote regions, certainly required the expenditure of work, and therefore the storage of a certain amount of energy which may perhaps need to be ascribed to the shell. However, this energy of creation of the shell with compensating charges does not come into the picture because only the changes which occur after the creation of the shell are considered in the calculations.

But, in moving the compensating charges about, local microscopic electric and magnetic fields about them will change, and it is possible that a changing surface distribution of energy will need to be assigned to these moving compensating charges.

It turns out that equally self-consistent but different Poynting vectors are obtained with various assumptions as to surface energy densities to be assigned to the shell and compensating charges. We begin by assuming that zero surface energy needs to be assigned to the shell.

4. The charges through whose motion the energy flow is defined can be regarded as making up a net surface charge density σ and a net surface current density i_1 given for a time by

$$\sigma = -(1/4\pi)D\cdot n, \tag{5}$$

$$i_1 = (c/4\pi)[H\times n], \tag{6}$$

where n is the unit vector outwardly normal to the shell. These equations take care of the transition to zero value of normal D, and tangential H. Equations (5) and (6) can continue to hold for some time without introduction of fresh charge. We see this by calculating the flux of current out of a closed curve in the surface. This is given by

$$\begin{aligned}\int_0 -[i_1\times n]\cdot ds &= -\frac{c}{4\pi}\int_0 [H\times n]\times n\cdot ds \\ &= \frac{c}{4\pi}\int_0 H\cdot ds \\ &= -\frac{c}{4\pi}\int\int \operatorname{curl} H\cdot dS \\ &= \int\int I_t\cdot dS \\ &= \int\int I_c\cdot dS - \int\int \frac{\partial\sigma}{\partial t} n\cdot dS, \end{aligned} \tag{7}$$

where I_c is the conduction current density.

The magnetic induction B enters the magnetic shell, and is completed as a solenoidal vector there by a sheet of induction parallel to the surface. It distributes itself there so as to take care of the transition of tangential E to zero. That is, if $\partial\Phi/\partial t$ is the rate of change of the surface flux sheet,

$$\partial\Phi/\partial t = -c[E\times n]. \tag{8}$$

Equation (8) is consistent with B in the volume, since the flux of $\partial\Phi/\partial t$ out of any closed curve in the surface is

$$\begin{aligned}\int_0 -\left[\frac{\partial\Phi}{\partial t}\times n\right]\cdot dS &= c\int_0 [E\times n]\times n\cdot ds \\ &= -c\int E\cdot ds \\ &= -c\int\int \operatorname{curl} E\cdot dS \\ &= \int\int \frac{\partial B}{\partial t}\cdot dS. \end{aligned} \tag{9}$$

5. According to Eq. (6) the rate of doing work by the moving charges is

$$\begin{aligned}\int\int E\cdot i_1 n\cdot dS &= \frac{c}{4\pi}\int\int E\cdot[H\times n]n\cdot ds \\ &= \frac{c}{4\pi}\int\int [E\times H]\cdot dS. \end{aligned} \tag{10}$$

We assume that this all comes from energy changes and activities within the volume alone. Then the Poynting vector $P_1 = (c/4\pi)[E\times H]$ may be regarded as the energy flow density out of the region.

6. The conclusion drawn in the preceding paragraph rests on the assumption that no changes occur in the energy which may need to be regarded as stored in the compensating

charges, the shell, or the inner magnetic shell. This assumption leads to a self-consistent energy flow vector.

For suppose a large volume is subdivided into small volumes, and each volume as well as the whole is surrounded by a shell. For each small volume we correlate the changes of energy with the activity of the adjacent charges in its shell. But the activities of the charges adjacent, respectively, in the same shell to two contiguous volumes will be equal and opposite. Hence if the activities of the charges in all the shell are added up, we will be left only with the activities of the charges in the outer shell. Hence the energy change in the whole large volume, as defined by the activity of the charges in the outer shell, is equal to the sum of the energy changes in the smaller volumes, as defined by the activities of the charges in their respective shells, since no energy is assigned to the charges in the shells themselves. Thus the energy changes in a given total volume as defined by activities of charges in shells subdividing it are independent of the particular way in which the subdividing shells may be constructed.

7. Instead of moving charges directly along lines of current flow as defined by (6), and getting work at a rate per cm² given by

$$E\cdot i_1=(c/4\pi)[E\times H]\cdot n,$$

we may move charges in other ways and get other expressions for the activity per cm². For example, if charges are moved about in a small circular path in the shell so as to make a current of magnitude i flowing in a small circle, they will produce the same magnetic effect as a magnetic double layer of intensity $M=\frac{1}{2}i$ per cm² over the circle. At the same time, the rate of doing work upon the charges will be

$$(1/c)(\partial Bn/\partial t)i=M(\partial Bn/\partial t)\text{ ergs/sec. per cm}^2,$$

of the circle.

Let M be an arbitrary scalar function of position in space. Let charges be moved about in small circles in the shell so as to produce the same magnetic effects as a double layer in the shell of moment M per cm². Then H will have a discontinuity at the double layer of an amount equal to $4\pi\,\text{grad}_n M$. The additional current sheet i_2 necessary to take care of the transition for tangential H to zero will now be

$$i_2=\frac{c}{4\pi}[(H-4\pi\,\text{grad}\,M)\times n]. \qquad (11)$$

The rate of doing work upon the charges, per unit area, is now

$$E\cdot i_2+M\frac{\partial B_n}{\partial t}$$

$$=\left\{\frac{c}{4\pi}[E\times H]-c[E\times\text{grad}\,M]+M\frac{\partial B}{H}\right\}\cdot n,$$

$$=\left\{\frac{c}{4\pi}[E\times H]-c\,\text{curl}\,ME\right\}. \qquad (12)$$

We thus arrive at the alternative Poynting vector

$$P_2=\frac{c}{4\pi}[E\times(H-4\pi\,\text{grad}\,M)]$$
$$+M\frac{\partial B}{\partial t}=\frac{c}{4\pi}[E\times H]-c\,\text{curl}\,ME. \qquad (13)$$

It evidently has the same divergence as P_1.

8. We also might move the compensating charges about in the shell as follows. Let N be an arbitrary scalar function of position. Over the inner surface of the shell lay electric double layers of moment N per cm²—that is, maintained by impressed normal electromotive force of magnitude $N/4\pi$. Let the moving charges rise up through these double layers, with a current density I_t. The rate of this working will be $-(1/4\pi)NI_t\cdot n$ per unit area. Above these double layers the compensating current will be again given by Eq. (6). However, tangential E will now be changed to $E_t-1/4\pi\,\text{grad}_t N$. The rate of working upon the changes per unit area now is

$$\left[E-\frac{1}{4\pi}\text{grad}\,N\right]\cdot i_1-\frac{1}{4\pi}NI_t\cdot n$$

$$=\frac{c}{4\pi}\left\{\left[\left(E-\frac{1}{4\pi}\text{grad}\,N\right)\times H\right]-\frac{1}{c}NI_t\right\}\cdot n$$

$$=\frac{c}{4\pi}\left\{[E\times H]-\frac{1}{4\pi}[\text{grad}\,N\times H]-\frac{1}{c}NI_t\right\}\cdot n$$

$$=\frac{c}{4\pi}\left\{[E\times H]-\frac{1}{4\pi}\,\text{curl}\,NH\right\}\cdot n. \qquad (14)$$

We thus arrive at the alternative Poynting vector

$$P_3=\frac{c}{4\pi}\left\{\left[\left(E-\frac{1}{4\pi}\text{grad } N\right)\times H\right]-\frac{1}{c}NI_t\right\}$$

$$=\frac{c}{4\pi}\left\{[E\times H]-\frac{1}{4\pi}\text{curl } NH\right\}. \quad (15)$$

It evidently has the same divergence as P_1.

9. It is frequently stated[1] that the conventional Poynting vector P_1 differs from all alternatives such as P_2 and P_3 in that P_1 is a function of the field vectors alone. This is not true. In (13) and (15) we may let M or N be scalar functions of the field vectors such as a, $bE\cdot H$, etc., where, of course, a and b are dimensional constants.

10. We call particular attention to the form which P_3 takes if we set

$$-(1/4\pi)N=\varphi, \quad (16)$$

where φ is a scalar potential, and A a corresponding vector potential of the electromagnetic field. That is

$$E=-\text{grad } \varphi-1\partial A/c\partial t, \quad (17)$$

$$B=\text{curl } A. \quad (18)$$

We then get

$$P_3=\varphi I_t-(1/4\pi)[(\partial A/\partial t)\times H]. \quad (19)$$

The first term φI_t is the formula for energy flow used extensively by electrical engineers of power systems, and where displacement currents are only a small part of the total current, may be determined directly by wattmeters. We see that to make it valid and have the same divergence as the Poynting vector, the second term $-(1/4\pi)\times[(\partial A/\partial t)\times H]$ must be added.

11. In Section 6 we have seen that by assigning all the activity of the compensating charges in the shell space to energy changes within the enclosed volume and assigning no energy changes to the shell and compensating charges themselves, a consistent definition of energy of the electromagnetic field may be obtained in that the energy assigned to a large region will be the sum of the energies assigned to all the sub-regions into which it may be divided, independently of the manner of the sub-division. We have already intimated that it is not necessary to be so drastic with respect to assignment of energy to the shell and compensating charges to get this result.

We may assign any energy density to the shell and compensating charges, provided that upon reversal of the direction of the normal to the shell the energy density assigned changes sign. In this case the argument of Section 6 goes through unchanged. At the same time, the mathematical expression for the energy density assigned to the shell and compensating charges may agree with the energy which will be assigned to charges and currents within the volume.

12. For example, in 4 we may assign an energy per unit area to the current sheet i_1 of Eq. (6) of an amount

$$T'=\frac{1}{c}A\cdot i_1 \quad (20)$$

where A is an arbitrary vector function of position. If T is then the appropriate energy density for the enclosed volume

$$-\frac{\partial}{\partial t}\int\int\int T\,dV$$

$$=\frac{\partial}{\partial t}\int\int T'n\cdot dS+\int\int\frac{c}{4\pi}[E\times H]\cdot dS$$

$$=\frac{\partial}{\partial t}\int\int\frac{1}{c}A\cdot\frac{c}{4\pi}[H\times n]n\cdot dS$$

$$+\int\int\frac{c}{4\pi}[E\times H]\cdot dS$$

$$=\int\int\frac{c}{4\pi}\left\{\frac{\partial}{\partial t}\left[\frac{1}{c}A\times H\right]+[E\times H]\right\}\cdot dS. \quad (21)$$

We thus arrive at a new alternative Poynting vector

$$P_4=\frac{c}{4\pi}\left\{[E\times H]+\frac{\partial}{\partial t}\left[\frac{1}{c}A\times H\right]\right\}$$

$$=\frac{c}{4\pi}\left\{\left[\left(E+\frac{1}{c}\frac{\partial A}{\partial t}\right)\times H\right]+\frac{1}{c}\left[A\times\frac{\partial H}{\partial t}\right]\right\}. \quad (22)$$

Its divergence differs from P_1, for

[1] For example, see Jeans, *Electricity and Magnetism*, fourth edition, p. 519.

$$-\operatorname{div} P_4 = -\operatorname{div} P_1 - \frac{1}{4\pi}\frac{\partial}{\partial t}\left(H\cdot\operatorname{curl} A - \frac{4\pi}{c}A\cdot I_i\right). \quad (23)$$

Letting T_1 be the energy density determined by the conventional Poynting vector P_1, we see that

$$T_4 = T_1 + \frac{1}{c}A\cdot I_i - \frac{1}{4\pi}H\cdot\operatorname{curl} A. \quad (24)$$

This equation does assign an energy $(1/c)A\cdot i_1$ per unit area, to the compensating charges, consistent with the initial hypothesis of this section.

13. In particular, we may let A be a vector potential of the electromagnetic field defined by Eqs. (17) and (18). Then (22) and (23) become

$$P_5 = \frac{c}{4\pi}\left\{-[\operatorname{grad}\varphi\times H] + \frac{1}{c}\left[A\times\frac{\partial H}{\partial t}\right]\right\}$$

$$= -\frac{c}{4\pi}\operatorname{curl}\varphi H + \varphi I_i + \frac{1}{4\pi}\left[A\times\frac{\partial H}{\partial t}\right]. \quad (25)$$

So far as its divergence is concerned, P_5 is equivalent to

$$P_6 = \Phi I_i + \frac{1}{4\pi}\left[A\times\frac{\partial H}{\partial t}\right] \quad (26)$$

and (24) becomes

$$T_6 = T_1 - \frac{H\cdot B}{4\pi} + \frac{1}{c}A\cdot I_i. \quad (27)$$

14. Again in Section 4, we may assign an energy per unit area to the charge sheet σ of Eq. (5) of an amount

$$T'' = \varphi\sigma \quad (28)$$

where φ is an arbitrary scalar function of position. If T is the appropriate energy density for the enclosed volume

$$-\frac{\partial}{\partial t}\int\int\int T\,dV$$

$$= \frac{\partial}{\partial t}\int\int T''n\cdot dS + \int\int\frac{c}{4\pi}[E\times H]\cdot dS$$

$$= \frac{\partial}{\partial t}\int\int -\frac{1}{4\pi}\varphi D\cdot dS + \int\int\frac{c}{4\pi}[E\times H]\cdot dS$$

$$= \int\int\left\{-\frac{1}{4\pi}\frac{\partial}{\partial t}(\varphi D) + \frac{c}{4\pi}[E\times H]\right\}\cdot dS. \quad (29)$$

We thus arrive at a new alternate Poynting vector

$$P_7 = \frac{c}{4\pi}[E\times H] - \frac{1}{4\pi}\frac{\partial}{\partial t}(\varphi D). \quad (30)$$

Taking the divergence of P_7, we find for the energy density

$$T_7 = T_1 + \frac{1}{4\pi}D\cdot\operatorname{grad}\varphi + \varphi\rho \quad (31)$$

where ρ is the charge density. Equation (31) assigns to the compensating charges the energy $\varphi\sigma$ per unit area, consistent with the initial hypothesis (28).

15. Now let φ be a scalar potential of the electromagnetic field. Then (30) becomes

$$P_7 = -\frac{c}{4\pi}\operatorname{curl}\varphi H + \varphi I_c - \frac{1}{4\pi}\frac{\partial\varphi}{\partial t}D - \frac{1}{4\pi}\left[\frac{\partial A}{\partial t}\times H\right], \quad (32)$$

where I_c is the conduction current density. This has the same divergence as

$$P_8 = \varphi I_c - \frac{1}{4\pi}\frac{\partial\varphi}{\partial t}D - \frac{1}{4\pi}\left[\frac{\partial A}{\partial t}\times H\right]. \quad (33)$$

16. We may illustrate P_6 and P_8 by a few examples. For steady fields

$$P_6 = P_8 = \varphi I_c. \quad (34)$$

They, therefore, show energy flowing only where there is conduction current, and do not show energy flowing in closed paths in empty space as P_1 does. For a stationary magnetic field, produced by steady conduction currents in otherwise empty space

$$T_6 = -\frac{H^2}{8\pi} + \frac{1}{c}A\cdot I_c. \quad (35)$$

The energy assigned to empty space is everywhere negative. Similarly for a stationary electric field produced by stationary charges in otherwise empty space

$$T_8 = -E^2/8\pi + \varphi\rho \quad (36)$$

and again, the energy assigned to empty space is negative.

Consider a plane electromagnetic wave traveling in the positive z directions and with E and H parallel to the x and y axes, respectively, so that

$$E_x = E_0 \sin 2\pi f(t - z/c), \tag{37}$$

$$H_y = H_0 \sin 2\pi f(t - z/c), \tag{38}$$

$$E_0 = H_0. \tag{39}$$

We may choose the scalar potential $\varphi = 0$ and for the vector potential which will have a component along the x axis only

$$A_x = \frac{c}{2\pi f} E_0 \cos 2\pi f\left(t - \frac{z}{c}\right). \tag{40}$$

The total current density will all be displacement current density and will have a component along the x axis only

$$I_t = \frac{f}{2} E_0 \cos 2\pi f\left(t - \frac{z}{c}\right). \tag{41}$$

For the conventional Poynting vector

$$P_1 = \frac{c}{4\pi} E_0^2 \sin^2 2\pi f\left(t - \frac{z}{c}\right), \tag{42}$$

$$T_1 = \frac{1}{4\pi} E_0^2 \sin^2 2\pi f\left(t - \frac{z}{c}\right). \tag{43}$$

For P_6

$$P_6 = \frac{c}{4\pi} E_0^2 \cos^2 2\pi f\left(t - \frac{z}{c}\right), \tag{44}$$

$$T_6 = \frac{1}{4\pi} E_0^2 \cos^2 2\pi f\left(t - \frac{z}{c}\right). \tag{45}$$

We see that P_6 like P_1 shows the wave as transporting its energy density with the propagation velocity c, but P_6 places its energy one quarter wave-length behind that of P_1.

Consider now the entirely valid energy flow vector

$$P_9 = \tfrac{1}{2}(P_1 + P_6). \tag{46}$$

Then in the example we are considering,

$$P_9 = \frac{c}{8\pi} E_0^2, \tag{47}$$

$$T_9 = \frac{1}{8\pi} E_0^2. \tag{48}$$

Thus all periodicity disappears in the energy flow and energy density distribution of the electromagnetic wave. However, as soon as the wave interacts with matter, the periodicity reappears in the energy. For example, suppose the wave is absorbed by a current sheet I_x at $z = 0$, so that

$$I_x = \frac{c}{4\pi} H_0 \sin 2\pi f t. \tag{49}$$

Then P_9 assigns an energy density to this current sheet of an amount $(1/2c)A_x I_x$ which varies in time according to equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2c} A_x I_x\right) = \frac{c}{8\pi} E_0^2 - \frac{c}{4\pi} E_0^2 \sin^2 2\pi f t. \tag{50}$$

Comparing with (47), we see then that there will be an appearance of energy as heat or otherwise given by

$$P_9 - \frac{\partial}{\partial t}\left(\frac{1}{2c} A_x I_x\right) = \frac{c}{4\pi} E_0^2 \sin^2 2\pi f t = P_1(z = 0). \tag{51}$$

The Use of Secondary Electron Emission to Obtain Trigger or Relay Action

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The use of secondary electrons to obtain trigger action similar to that of a thyratron is described. An experimental tube and the necessary circuits by which this action is achieved are discussed. This combination gives the features of a triode with a relay or on and off feature, resulting in an amplifier, oscillator, modulator, or other vacuum tube device which may be turned on or off abruptly at high or low frequencies. In addition, it can be used to replace thyratrons in many of their circuits where very low impedance is not necessary and is capable of much greater speeds of operation in such applications.

INTRODUCTION

DURING the past few years techniques have been developed which make practical the utilization of secondary electrons in commercial types of vacuum tubes. These developments have been applied primarily to the problem of the secondary electron multiplication of current. There is described herein another practical use of secondary electrons: namely, to obtain electronic trigger or relay action.

The negative resistance characteristic of an element that is emitting secondary electrons was discovered by A. W. Hull[1] in 1915 and is frequently referred to as the dynatron characteristic after the name given by him to his tube. This was never employed widely, however, because of its instability. Recognizing that this instability was due to contamination from the hot cathode reaching the secondary emitting surface, and that this contamination travelled in straight lines, Jonker and Overbeek[2] in 1938 described a practical and stable amplifier tube employing secondary emission. In their tube the secondary emitting surface was shielded from the cathode and the primary electrons were constrained to travel in curved paths in order to reach it. In the experimental tube described herein this same principle is employed to obtain stability of operation.

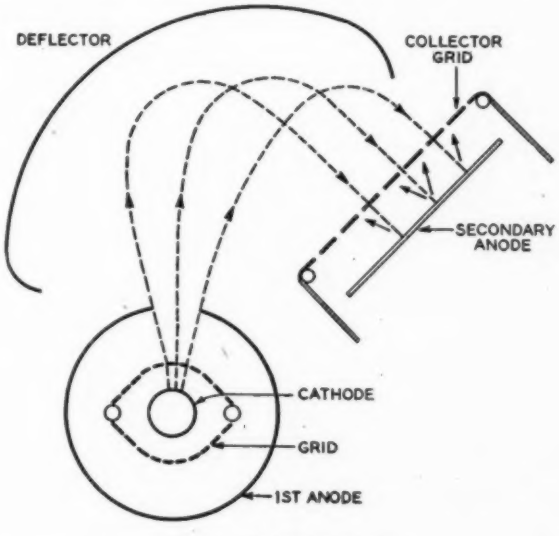

FIG. 1. Tube structure.

TUBE STRUCTURE

The tube structure is shown in sectional view in Fig. 1. Surrounding the cylindrical cathode, there is a control grid and coaxial cylindrical anode of the usual kind but with the modification that a hole or slot in the anode allows about 10 percent of the electrons to pass through. These electrons are deflected by the deflector and attracted by the collector grid so that they follow approximately the dotted paths shown. The deflector plate is connected to the cathode and the collector grid is maintained at a high positive potential. The shape of the deflector plate was determined by rubber model studies.[3]

The trigger action of the tube depends on the properties of the secondary anode which will be described further on. This element, bombarded by the primary electrons from the cathode, emits secondary electrons in a ratio greater than unity and these secondaries are drawn off by the col-

[1] A. W. Hull, Proc. I. R. E. **6**, 5 (1918).

[2] J. L. H. Jonker and A. J. W. M. v. Overbeek, W. Eng. **15**, 150 (1938).

[3] P. H. J. A. Kleynen, Philips Tech. Rev. [2] **11**, 338 (1937).

lector grid which is always positive with respect to the secondary anode.

Properties of the Secondary Anode

Figure 2 shows a family of current *vs.* potential characteristics of the secondary anode. The different curves correspond to different potentials on the collector grid. As the potential is increased the current passes through the zero axis twice. At each of these two points the number of primary electrons is equal to the number of secondaries which are drawn off of the surface. If the secondary anode is at the higher zero potential the external circuit may be entirely disconnected and the element will float in a perfectly stable manner at this voltage. At the lower zero potential, however, the equilibrium is unstable and if the external circuit is disconnected the potential will "fall" either way. If it falls down the element returns to zero and if it falls up it jumps to the second zero potential and floats stably there.

To understand the reason for this stability it is only necessary to consider the conditions affecting the addition or removal of charge from the secondary anode. Suppose that initially this element is floating at the second zero and that it starts to drift lower in potential. The voltage difference between it and the collector grid will increase and more secondaries will be drawn away from it. It will therefore go more positive in potential and will thus be brought back to equilibrium. Conversely if it starts to drift up in voltage the potential difference between it and the grid will be decreased and fewer secondaries will be drawn across, thereby decreasing its potential by the accumulation of more negative charge.

Similar reasoning will explain the instability of the first zero, except that, whereas for the second zero the number of secondaries which are effective is that drawn off by the potential difference between the collector grid and secondary anode, for the first zero all of the secondaries are drawn across and the true secondary to primary ratio is the important factor. The voltage of the first zero is therefore determined primarily by the secondary to primary ratio of the surface of the secondary anode and is thus approximately the same for all values of collector grid potential, but the voltage of the second zero is determined primarily by the geometry of the collector grid and secondary anode structure of the tube.

The curves of Fig. 2 were taken with the first anode current approximately constant at 4 milliamperes. It was adjusted to this value at the start of each curve and changed very little. The first anode was held at the potential of the collector grid.

If the secondary anode is connected to ground through a high resistance, electrons will flow

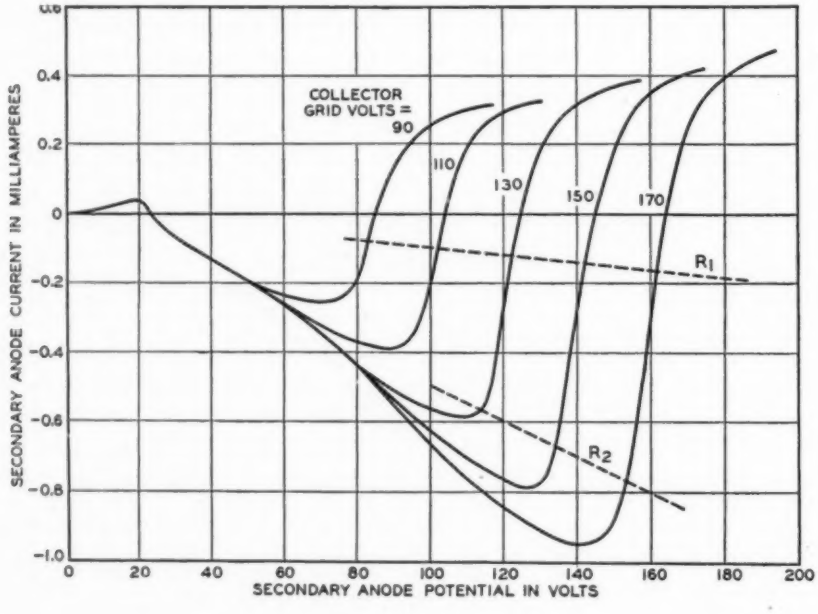

FIG. 2. Characteristics of secondary anode.

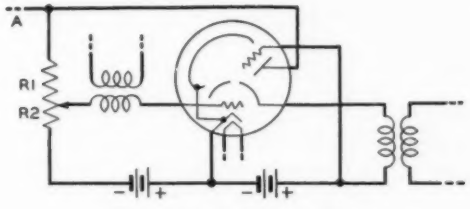

FIG. 3. Basic circuit No. 1.

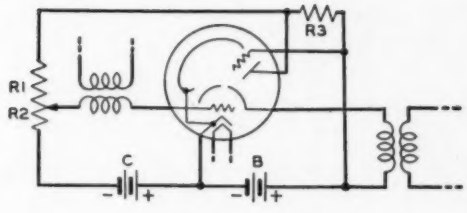

FIG. 4. Basic circuit No. 2.

from the ground through this resistance to it because of its positive potential and these electrons will counteract the effect of an equal number of secondaries simply by replacing the charge carried away by the latter. Now the effect of reducing the secondary to primary ratio is that the second zero point or stable floating potential is reduced. The secondary anode will now float at a potential slightly less than it did when free. This new floating potential is determined by the intersection of the I–V characteristic with the line whose slope is equal to the load resistance. For example, the dotted lines R_1 and R_2 of Fig. 2 have the slopes of one megohm and two-tenths megohm, respectively, and the potentials of the points at which they cross the curves are the floating potentials for these resistance values. Resistances smaller than about 1.4×10^5 ohms do not intersect the curves at all and the critical resistance is therefore near this value. It is given by the formula

$$R_c = E_m/I_m,$$

so that in general we have that

$$R > E_m/I_m, \qquad (1)$$

where E_m and I_m are the voltage and current of the negative maximum of the characteristic. The secondary anode will not float at values of resistance less than R_c for then the number of electrons supplied via the resistance will be in excess of the number of secondaries needed to maintain the effective secondary to primary ratio at a value equal to unity and the secondary anode will simply charge up negatively and its potential will be brought to zero.

CIRCUIT APPLICATIONS

These principles may be usefully applied to get on and off action in a vacuum tube similar to those obtained in a gas tube or thyratron. A simple circuit for accomplishing this is shown as circuit No. 1 on Fig. 3. The tube is shown in schematic form, a break in the first anode being indicative of the slot through which the electrons pass to the collector grid and finally to the secondary anode shown in the upper right quadrant of the tube diagram. The values of R_1 and R_2 are adjusted so that their total adds up to a value greater than R_c, preferably considerably greater. A value of from 1 to 5 megohms is recommended. Between the cathode and R_2 is connected a negative or C battery of a value sufficient just to cut off the flow of electrons from the cathode. The grid of the triode structure is connected as shown between R_1 and R_2. The first anode is connected to a positive battery in a conventional manner through a load impedance and the collector grid is maintained at the highest voltage of the battery.

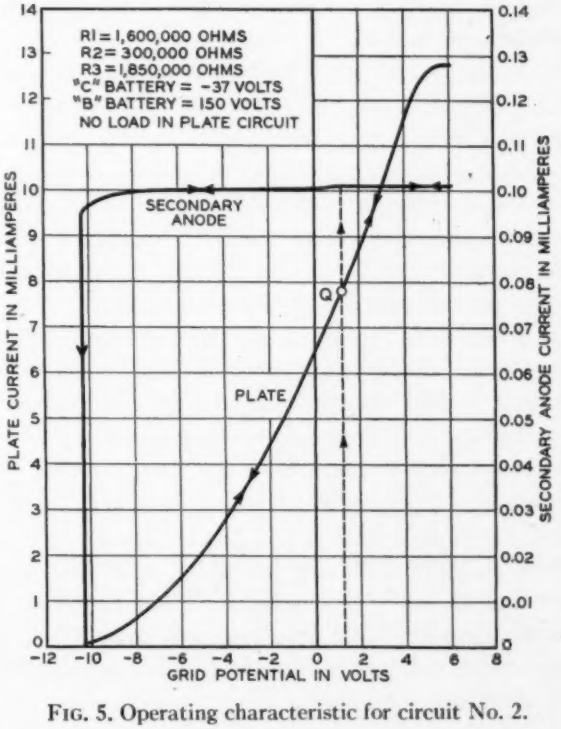

FIG. 5. Operating characteristic for circuit No. 2.

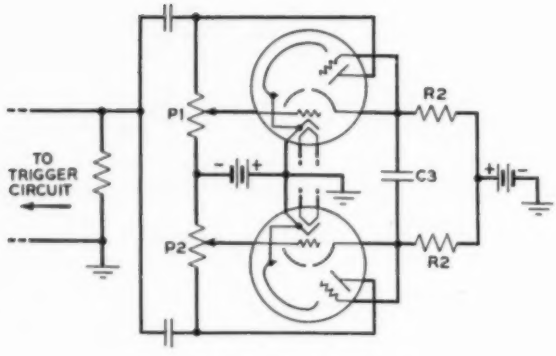

FIG. 6. Pulse inverter circuit.

Suppose that the cathode heater is turned on and brought to operating temperature. The secondary anode and the inner grid will all be at the negative C potential and no electrons can flow through the tube. If now the potential of the secondary anode is momentarily raised to a value greater than the first zero potential (Fig. 2) the potential of the grid will be raised a few volts because of the drop across R_1 and R_2 and a small electron current will flow through the tube. As soon as electrons flow to the secondary anode its potential will jump to the second zero voltage carrying the grid bias up to the operating point. The tube has then triggered on or "fired" and can be utilized as an amplifier through the transformers shown.

It can be triggered off by decreasing momentarily the potential of the secondary anode to a value slightly less than the first zero value or by sending a negative pulse into the input or grid circuit of sufficient value to cut off momentarily the electron flow. The triggering pulses to the secondary anode may be applied at A in the diagram through a suitable condenser.

Circuit No. 2 (Fig. 4) is the same as circuit No. 1 with the addition of R_3. R_1, R_2, and R_3 now form a bleeder resistance across the total supply battery maintaining the secondary anode at a potential slightly in excess of the first zero voltage when no electrons are flowing in the tube. R_1 and R_2 are so adjusted that, as in circuit No. 1, the first grid is held just beyond cut-off when the tube is in the off condition. This requires a slightly greater C battery than for circuit No. 1 since there will be a positive voltage drop through R_2 when the tube is off.

Now if, through the input transformer, a positive potential, which may be as little as a fraction of a volt, is impressed on the grid, enough current (less than a microampere) will be passed to trigger the tube on; the secondary anode will jump to the second zero as in the case of circuit No. 1. Circuit No. 2 thus has an advantage over circuit No. 1 in that considerably smaller voltages serve to trigger it on. It may be triggered off by applying a negative signal or pulse in the grid circuit great enough to cut off momentarily the electron stream.

Figure 5 shows an operating characteristic for circuit No. 2. A positive potential of 1.2 volt applied between the junction of R_1 and R_2 and the control grid triggers the tube on and the plate current jumps to point Q on the operating characteristic, i.e., from zero to 7.75 milliamperes. Positive or negative potentials may then be applied and the plate current will swing up and down as in an ordinary amplifier tube. If a negative potential greater than 10.3 volts is applied the tube will trigger off as shown. A comparison of the shape and slope of this curve with those for the triode structure alone (similar to a Western Electric 244-A) confirmed the analytically determined conclusion that the addition of the trigger mechanism has very little effect on the operating properties of the triode structure. This is mainly because the secondary anode is so well shielded from the main electron steam.

The value of R_3 may be obtained by calculating the total value of R_1 plus R_2 as for circuit No. 1 and then choosing a value for R_3 such that the junction of R_1 and R_3 will have a potential with no electrons flowing in the tube ten volts or so greater than that of the first zero of the characteristic (Fig. 2). Obviously R_3 need not be connected to the highest potential of the power supply although in general it probably will be simplest to do so for circuit reasons. The off and on potentials of the inner grid and the amount of its swing may be adjusted by varying the relative values of R_1 and R_2 and the voltage of the C battery.

The tube in circuit No. 2 is equivalent in most respects to a hot cathode thyratron. These differences may be noted, however. Since there is no deionization time limitation the vacuum tube may be made many times faster than the thyra-

tron. The vacuum tube may be "extinguished" or triggered off by a negative voltage on the grid and while the tube is on, it is not noisy like the thyratron and its grid maintains complete control over the space current for amplification oscillation, detection, modulation, etc. Against these advantages should be noted the fact that the vacuum tube for the same size cathode has a higher internal impedance than the gas tube has, or in other words, a relatively larger cathode must be used in the vacuum tube to obtain the low impedance inherent in the gas tube where this characteristic is important. Thus the gas tube may be replaced in many of its applications by the vacuum tube in circuit No. 2.

Two typical gas tube circuits, a pulse inverter or scale of two counter circuit and a relaxation oscillator, are shown as examples in Figs. 6 and 7, with the vacuum tube replacing the gas tube. The pulse inverter shown makes use of circuit No. 1 and the oscillator circuit No. 2. In both of these circuits the collector grid is shown connected to the first anode. It may also be connected directly to the high side of the battery as shown in the circuits above.

In the pulse inverter, the positive trigger pulses fire the two tubes alternately. In the relaxation oscillator the condenser C is charged through R_4 until the first anode potential is great enough to fire the tube. When the tube fires it discharges the condenser and starts a new cycle of operation.

In the conventional two-tube multivibrator circuit the plate of one tube is in phase with the grid of the other and is therefore coupled to it to give positive feed-back action. In the secondary emission tube shown herein the secondary anode is in phase with the control grid so that by coupling them together with a condenser a one-tube multivibrator may be obtained. This circuit is shown on Fig. 8. In operation the grid and secondary anode potentials follow wave forms similar to those of a multivibrator. The

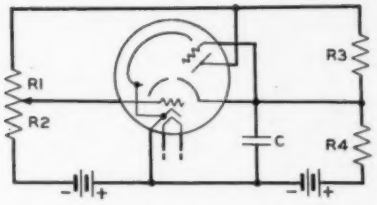

FIG. 7. Relaxation oscillator.

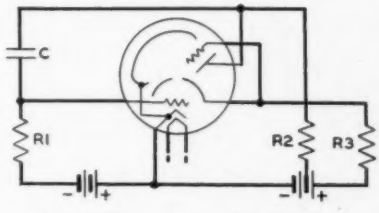

FIG. 8. One-tube multivibrator.

negative grid bias is adjusted to a value that would be suitable for amplification. Out-of-phase pulses may be obtained across R_3 or it may be shorted out since it is not essential to the multivibrator action. R_2 should be less than R_c in order that operation may be obtained on the negative resistance portion of the characteristic of the secondary anode.

Variations in Tube Design

Although the tubes described herein have been made up with triode structures, it is obvious that they may be made up with tetrode or pentode structures since the triggering device is external to the plate of the amplifier section of the tube.

For faster operation the opening in the first anode or plate may be increased so that half of the electron stream is utilized for the triggering function. This gives a plate impedance about twice as great for the amplifier section.

If a separate lead is brought out for the deflector, this element may be used as an independent control of the triggering action. A sufficient negative bias on the deflector will prevent electrons from passing through the slot in the plate.

Errata: Electrostatic Electron Lenses with a Minimum of Spherical Aberration

[J. App. Phys. 13, 49 (1942)]

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(Received June 10, 1942)

IN the recent paper by the above title, the percentages given in the third and fourth paragraphs of the "note added in proof" are incorrect. The correct values are as follows:

For lenses of equal focal lengths and equal distances between principal planes, the spherical aberration of (14) is 118.5 percent of that of (2); and the spherical aberration of (10) is 11.45 percent of that of (2). This agrees with the result that the weak symmetrical lens with minimum spherical aberration is (2). The weak immersion lens with minimum spherical aberration is[1]

$$V(z)=A\int_{-\infty}^{z}\exp(-z^2)dz; \qquad \text{(A)}$$

the spherical aberration of this lens is 11.11 percent of that of (2), only slightly better than that of (10).

For lenses of equal focal lengths and equal "thicknesses" ("thickness" being defined according to the method discussed in the paper), the spherical aberration of (14) is 82.4 percent of that of (2); and for (10) it is 40.5 percent of that of (2). If the definition is changed, as before, to read 1/10 and 9/10 instead of 1/100 and 99/100, the spherical aberration of (14) is 94.1 percent of that of (2); and for (10) it is 25.0 percent of that of (2).

Using criteria, other than Scherzer's, for comparing lenses, it is, of course, possible to construct "better" lenses than the lens (2). Since it is uncertain which criterion would be best, the choice between (2) and (14) would have to be made experimentally. It was shown in detail in the paper that, in principle, these lenses could be constructed rather easily.

I am indebted to Dr. E. G. Ramberg of the RCA Manufacturing Company, Inc., and to Mr. R. G. Hutter and Dr. L. Marton of Stanford University, who have kindly called my attention to the errors reported here.

[1] Rebsch and Schneider, Zeits. f. Physik **107**, 138 (1937).

New Instrument Booklets

A new stock catalog, No. 18, has recently been issued by the Ohmite Manufacturing Company, 4835 Flournoy Street, Chicago, Illinois. Rheostats, resistors, tap switches, chokes, and attenuators are listed in this 16-page illustrated booklet.

Bulletin No. Z6000, superseding Bulletin No. Z5000, provides a condensed listing of the principal measuring and control instruments offered by the Wheelco Instruments Company, Harrison and Peoria Streets, Chicago, Illinois. Notice is given that aluminum casings on all instruments have been replaced by steel or cast iron cases, for delivery after July 1, 1942.

Direct current and alternating current indicating instruments for mounting on switchboards are described in a new 12-page catalog, No. 4220, published by the Roller-Smith Company, Bethlehem, Pennsylvania. These instruments include d.c. ammeters and voltmeters and a.c. ammeters, voltmeters, wattmeters, frequency and power factor meters, synchroscopes, rectangular triplex ammeters, and horizontal edgewise triplex ammeters. Round surface mounting models and square surface or flush models are listed and described.

Catalog T-625 entitled "The Homo Method for Tempering" has just been released by the Leeds & Northrup Company, 4934 Stenton Avenue, Philadelphia, Pennsylvania. This 28-page, well-illustrated booklet describes various uses of the L & N Homo furnace and contains new data and new charts.

An 8-page catalog giving technical data on Emby photoelectric cells has been received from the Emby Products Company, 1800 West Pico Boulevard, Los Angeles, California. A price list is included.

A new 12-page booklet, giving the history, operating characteristics, and details of construction of its mass spectrometer, is announced by the Westinghouse Electric and Manufacturing Company. Performance data with mass spectra for different gases are also given. A copy of technical data 85–940 may be secured from Department 7-N-20, Westinghouse Electric and Manufacturing Company, East Pittsburgh, Pennsylvania.